
Quantum Conditional Stochastic Processes

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Quantum mechanics contains certain novel mathematical concepts. Among these are complex numbers, Hilbert spaces with their unitary and self-adjoint operators, states represented by complex vectors, superpositions of states, collapse of wave functions, Born's rule for probabilities and others. If we accept that quantum mechanics is probabilistic, then these concepts can be derived and they become secondary. In this work, we begin with what we call a *conditional stochastic process*, which is based on real numbers and probabilities. As we shall see, such processes are defined by three simple axioms. We then use conditional stochastic processes to derive quantum mechanics by employing a correspondence called a *dictionary*. We also show that the converse holds. That is, beginning with a quantum system, we employ the dictionary to derive a conditional stochastic process.

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1 Introduction


Recently there have appeared a considerable number of papers on *conditional stochastic processes* and their relationship to quantum mechanics [1–9]. The most mathematically complete is [6], but we shall follow the interesting papers [1, 3]. However, our work is simpler and more

general. It is simpler in the sense that fewer parameters are involved and more general in the sense that it includes an infinite number of configurations and separable infinite dimensional Hilbert spaces. Such Hilbert spaces are general enough to include most of the work on quantum mechanics.

Until 1925, physics involved directly observable quantities such as position, momentum, energy, forces and electromagnetic fields. The main mathematical tool used for solving problems was differential equations. With the advent of quantum mechanics [10, 11], a number of novel mathematical entities seemed to be necessary for physical descriptions. A complex Hilbert space \mathcal{H} appeared, states represented by vectors in \mathcal{H} , observables given by self-adjoint operators on \mathcal{H} , dynamics given by unitary operators, complex superpositions of states and state collapse upon measurement, Born's probability rule, etc. Where did all these things come from? The main object of this work is to show that they come from a semi-classical real stochastic process satisfying a few basic axioms. We also show that the converse holds. That is, given a quantum system, we can produce such a stochastic process that generates it.

2 Conditional Stochastic Processes

A *conditional stochastic process* (CSP) is a triple (K, p, μ) that satisfies the following conditions. We require that $K = \{1, 2, \dots\}$ is a countable set of *configurations* and μ is a nontrivial probability measure on K so that $\mu(j) \neq 0$ for all $j \in K$. Thus, $0 < \mu(j) \leq 1$ for all $j \in K$ and

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$\sum_{j=1}^{\infty} \mu(j) = 1$. We also implicitly assume that this includes the special case in which $K = \{1, 2, \dots, N\}$ is finite. The conditional probability map $p: K \times \mathbb{R} \times K \rightarrow [0, 1] \subseteq \mathbb{R}$ is denoted by $p(j, t | k) = p(j, t, k)$ where t is time. We call $p(j, t | k)$ the probability that the system is at configuration $j \in K$ at time t given it is at $k \in K$ at initial time 0. We require that p satisfies the normalization conditions

$$\sum_{j=1}^{\infty} p(j, t | k) = \sum_{k=1}^{\infty} p(j, t | k) = 1 \quad (1)$$

as well as the trivialization condition

$$p(j, 0 | k) = \delta_{jk} \quad (2)$$

for all $j, k \in K$, where δ_{jk} is the Kronecker delta function. The system's standalone probability distribution $p_t: K \times \mathbb{R} \rightarrow [0, 1]$ is

$$p_t(j) = \sum_{k=1}^{\infty} p(j, t | k) \mu(k) \quad (3)$$

for every $j \in K, t \in \mathbb{R}$. By (1) we have that p_t is a probability distribution because

$$\begin{aligned} \sum_{j=1}^{\infty} p_t(j) &= \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} p(j, t | k) \mu(k) \\ &= \sum_{k=1}^{\infty} \mu(k) \sum_{j=1}^{\infty} p(j, t | k) \\ &= \sum_{k=1}^{\infty} \mu(k) = 1 \end{aligned}$$

Notice, by (2) we have

$$p_0(k) = \sum_{r=1}^{\infty} p(k, 0 | r) \mu(r) = \sum_{r=1}^{\infty} \delta_{kr} \mu(r) = \mu(k)$$

so $\mu(k)$ is the probability the system is initially at configuration k . We can write (3) as

$$p_t(j) = \sum_{k=1}^{\infty} p(j, t | k) p_0(k)$$

for all $j \in K, t \in \mathbb{R}$ which is called the law of total probability. In summary, a CSP is a triple (K, p, μ) satisfying (1) and (2). It is a simple probabilistic structure based on the real numbers.

A matrix of the form $U_t(j, k) = p(j, t | k)^{1/2}$ with indices $j, k \in K$ is the dynamical map and is supposed to describe the time evolution of the system. If $f: K \rightarrow \mathbb{R}$ is a random variable we form the column vector $|f\rangle$ whose

j th entry is $f(j)$. Then the time evolution of f is given by the vector $|U_t f\rangle$ where $U_t f$ is the matrix product of U_t and $|f\rangle$. But this is very limited since we only have this one dynamical map and there should be various ways a system evolves. This is where the complex numbers enter the situation! If we include complex numbers we have infinitely many possible dynamical maps! Letting \mathbb{C} be the set of complex numbers and i the imaginary unit, if $\theta: \mathbb{R} \times K^2 \rightarrow \mathbb{R}$ where $\theta(0, j, k) = 0$, we define a general dynamic map as a matrix with entries j, k :

$$U_t(j, k) = e^{i\theta(t, j, k)} p(j, t | k)^{1/2} \in \mathbb{C} \quad (4)$$

Since by (1) we have

$$\begin{aligned} \sum_{j=1}^{\infty} |U_t(j, k)|^2 &= \sum_{j=1}^{\infty} p(j, t | k) = \sum_{k=1}^{\infty} p(j, t | k) \\ &= \sum_{k=1}^{\infty} |U_t(j, k)|^2 = 1 \end{aligned}$$

so U_t is a unitary matrix ($U_t^* U_t = U_t U_t^* = I$ with U_t^* the conjugate transpose of U_t). This is precisely the type of evolutionary operators in quantum mechanics. We also have by (2) that

$$U_0(j, k) = e^{i\theta(0, j, k)} p(j, 0, k)^{1/2} = e^{i\theta(0, j, k)} \delta_{jk} = \delta_{jk}$$

so the matrix $U_0 = I$. We have seen where the complex numbers come from and now consider the Hilbert space.

3 The Dictionary

Let $\mathcal{H} = L_2(K, \mu)$ be the separable complex Hilbert space consisting of vectors $|\psi\rangle = (\psi_1, \psi_2, \dots)$ where $\psi_j \in \mathbb{C}$ with squared norm

$$\|\psi\|^2 = \sum_{j=1}^{\infty} |\psi_j|^2 \mu(j) < \infty$$

and inner product $\langle \psi, \phi \rangle = \sum_{j=1}^{\infty} \bar{\psi}_j \phi_j \mu(j)$. We call the orthonormal basis on \mathcal{H} given by

$$e_j = \left(0, 0, \dots, \frac{1}{\mu(j)^{1/2}}, 0, \dots \right)$$

where $\frac{1}{\mu(j)^{1/2}}$ is in the j th position, the configuration basis. The rank-one projection operators

$$P_j = \text{diag}(0, 0, \dots, 1, 0, \dots) = |e_j\rangle\langle e_j|$$

where 1 is in the j th position, are called the configuration projections. These operators form a projection valued

measure (PVM) and satisfy $P_j P_k = \delta_{jk} P_j$ for all $j, k \in K$ and $\sum_{j=1}^{\infty} P_j = I$. Letting tr be the trace, we have

$$\begin{aligned} \text{tr}(U_t^* P_j U_t P_k) &= \sum_{r=1}^{\infty} \langle e_r, U_t^* P_j U_t P_k e_r \rangle \\ &= \langle e_k, U_t^* P_j U_t e_k \rangle \\ &= \langle U_t e_k, |e_j\rangle \langle e_j| U_t e_k \rangle = \left| \langle e_j, U_t e_k \rangle \right|^2 \\ &= |U_t(j, k)|^2 = p(j, t | k) \end{aligned}$$

The dictionary [1, 3] given by

$$p(j, t | k) = \text{tr}(U_t^* P_j U_t P_k) \quad (5)$$

for all $j, k \in K, t \in \mathbb{R}$, translates the CSP as expressed by the left-hand side of (5) and the representation as expressed by the right-hand side of (5). The standalone probability becomes

$$p_t(j) = \sum_{k=1}^{\infty} p(j, t | k) \mu(k) = \text{tr} \left(U_t^* P_j U_t \sum_{k=1}^{\infty} \mu(k) P_k \right) \quad (6)$$

We have thus shown that a CSP describes a complex separable Hilbert space and its concepts are represented by the dictionaries (5), (6). We now show that the converse holds. In this way a quantum system describes a CSP.

Let \mathcal{H} be a separable complex Hilbert space. Since two separable Hilbert spaces of the same dimension are isomorphic, we can assume that $\mathcal{H} = L_2(K, \mu)$ where $K = \{1, 2, \dots\}$ and $0 < \mu(j) \leq 1$ with $\sum_{j=1}^{\infty} \mu(j) = 1$. Let $U_t, t \in \mathbb{R}$ be unitary operators on \mathcal{H} with $U_0 = I$ and let P_j be the rank-one projections on H defined previously. Define $p: K \times \mathbb{R} \times K \rightarrow [0, 1]$ as in the dictionary (5) by

$$p(j, t | k) = |U_t(j, k)|^2 = \text{tr}(U_t^* P_j U_t P_k)$$

and consider the triple (K, p, μ) . Since U_t is unitary we have for all $j \in K$ that

$$\sum_{k=1}^{\infty} p(j, t | k) = \sum_{k=1}^{\infty} |U_t(j, k)|^2 = 1$$

and for all $k \in K$ that

$$\sum_{j=1}^{\infty} p(j, t | k) = \sum_{j=1}^{\infty} |U_t(j, k)|^2 = 1$$

so (1) holds. Also, we have

$$p(j, 0 | k) = |U_0(j, k)|^2 = \delta_{jk}$$

for all $j, k \in K$ so (2) holds. We conclude that (K, p, μ) is a CSP so the converse holds.

Let (K, p, μ) be a CSP and let $\mathcal{H} = L_2(K, \mu)$ be the corresponding quantum system. We now show that various quantum concepts on \mathcal{H} are automatically described by (K, p, μ) . For example, most quantum systems have dynamics given by unitary operators $\widehat{U}_t = e^{it\widehat{H}}$ where \widehat{H} is a self-adjoint operator on \mathcal{H} called the *Hamiltonian*. Assuming that \widehat{H} has pure point spectrum with eigenvalues $\lambda_j \in \mathbb{R}$ so that $\widehat{H}e_j = \lambda_j e_j$, then $\widehat{U}_t e_j = e^{it\lambda_j} e_j, j = 1, 2, \dots$. Forming the corresponding CSP we have that $p(j, t | k) = \delta_{jk}$ and $U_t(j, k) = e^{it\lambda_j} \delta_{jk}$. In general, if \widehat{A} is a self-adjoint operator on \mathcal{H} with pure point spectrum λ_j and $\widehat{A}e_j = \lambda_j e_j$, then we can represent \widehat{A} on the corresponding CSP by the random variable $A(j) = \lambda_j, j = 1, 2, \dots$. Then A has *average value* at time 0 given by

$$\text{Ave}_0 = \sum_{j=1}^{\infty} \lambda_j p_0(j) = \sum_{j=1}^{\infty} \lambda_j \mu(j)$$

which is the usual quantum mechanical value. Conversely, if $A(j) = \lambda_j, j = 1, 2, \dots$, is a random variable on a CSP, then $\widehat{A} = \sum_{j=1}^{\infty} \lambda_j P_j$ is a self-adjoint operator on $L_2(K, \mu)$.

We refer to our references [1–7] which show how essentially all the quantum mechanical concepts are reproduced using the dictionary.

4 Induced States and Tensor Products

In this section we show that a CSP induces certain states on its corresponding quantum system. We also show that the natural product of two CSP's generates the tensor product of their corresponding quantum systems.

Let (K, p, μ) be a CSP and $L_2(K, \mu)$ be the corresponding quantum system. As we have seen these systems are related by the dictionary (5) and the standalone probability (6). We call

$$\rho_t = \sum_{k=1}^{\infty} \mu(k) U_t P_k U_t^*$$

an *induced state* of the system $L_2(K, \mu)$ at time t . In this way, different functions $\theta(t, j, k)$ induce various quantum states on $L_2(K, \mu)$. Also, notice that $p_t(j) = \text{tr}(\rho_t P_j)$ for all $t \in \mathbb{R}, j \in K$. The following results show that the dictionary can be written in a simpler way and that induced states have a usual form.

Theorem 1. (a) For all $t \in \mathbb{R}, k \in K$,

$$U_t P_k U_t^* = |U_t e_k\rangle\langle U_t e_k| = P_{U_t e_k}$$

is a one-dimensional projection so it is a pure quantum state.

(b) An induced state ρ_t is a trace 1 positive operation (a density operator) so it is a usual quantum state. Also, $\mu(k)$ is an eigenvalue of ρ_t with corresponding eigenvector $U_t e_k$.

(c) A dictionary can be written:

$$p(j, t | k) = \left| \langle e_j, U_t e_k \rangle \right|^2$$

and we have

$$p_t(j) = \sum_{k=1}^{\infty} \mu(k) \left| \langle e_j, U_t e_k \rangle \right|^2$$

Proof. (a) For all $t \in \mathbb{R}, k \in K$ we have

$$U_t P_k U_t^* = U_t |e_k\rangle\langle e_k| U_t^* = |U_t e_k\rangle\langle U_t e_k| = P_{U_t e_k}$$

which is a one-dimensional projection.

(b) By (a) we have $\rho_t = \sum_{k=1}^{\infty} \mu(k) P_{U_t e_k}$ so ρ_t is a positive operator with $\text{tr}(\rho_t) = \sum_{k=1}^{\infty} \mu(k) = 1$. Also $\mu(k)$ is an eigenvalue of ρ_t with corresponding eigenvector $U_t e_k$ for all $t \in \mathbb{R}, k \in K$.

(c) By (a) and (5) we have

$$\begin{aligned} p(j, t | k) &= \text{tr}(U_t^* P_j U_t P_k) = \text{tr}(P_j U_t P_k U_t^*) \\ &= \langle e_j, U_t P_k U_t^* e_j \rangle = \langle e_j, U_t | e_k \rangle \langle e_k | U_t^* e_j \rangle \\ &= \langle e_j, U_t e_k \rangle \langle U_t e_k, e_j \rangle = \left| \langle e_j, U_t e_k \rangle \right|^2 \end{aligned}$$

Finally (6) gives

$$p_t(j) = \sum_{k=1}^{\infty} \mu(k) p(j, t | k) = \sum_{k=1}^{\infty} \mu(k) \left| \langle e_j, U_t e_k \rangle \right|^2 \quad \square$$

We define

$$p_t(\rho_j) = p_t(j) = \text{tr}(\rho_t P_j) = \langle e_j, \rho_t e_j \rangle$$

This is a special case of *Born's rule*. Extending this by linearity, if $\psi = \sum_{j=1}^{\infty} c_j e_j$ is a unit vector, then

$$p_t(|\psi\rangle\langle\psi|) = \text{tr}(\rho_t |\psi\rangle\langle\psi|) = \langle \psi, \rho_t \psi \rangle$$

A *quantum effect* is a positive operator E on $L_2(K, \mu)$ with $\text{tr}(E) = 1$. An effect is a general quantum event. Extending our previous result by linearity, we have that if E is an effect, then $p_t(E) = \text{tr}(\rho_t E)$. This is a general Born's rule for the state ρ_t .

We now consider tensor products of Hilbert spaces and show that they occur naturally in this framework. Let (K_1, p_1, μ_1) and (K_2, p_2, μ_2) be CSP's and form the triple $(K_1 \times K_2, p_1 \times p_2, \mu_1 \times \mu_2)$ where

$$p_1 \times p_2(j_1, j_2, t | k_1 \times k_2) = p_1(j_1, t | k_1) p_2(j_2, t | k_2)$$

and

$$\mu_1 \times \mu_2(j_1 \times j_2) = \mu_1(j_1) \mu_2(j_2)$$

Theorem 2. (a) The triple $(K_1 \times K_2, p_1 \times p_2, \mu_1 \times \mu_2)$ is a CSP.

(b) The quantum system for (a) is the tensor product $L_2(K_1, \mu_1) \otimes L_2(K_2, \mu_2)$.

Proof. (a) For every $t \in \mathbb{R}$, $k_1 \in K_1$, $k_2 \in K_2$ we have

$$\sum_{j_1, j_2=1}^{\infty} p_1 \times p_2(j_1 \times j_2, t | k_1 \times k_2) = \sum_{j_1=1}^{\infty} p_1(j_1, t | k_1) \sum_{j_2=1}^{\infty} p_2(j_2, t | k_2) = 1$$

and similarly for every $t \in \mathbb{R}$, $j_1 \in K_1$, $j_2 \in K_2$ we have

$$\sum_{k_1, k_2=1}^{\infty} p_1 \times p_2(j_1 \times j_2, t | k_1 \times k_2) = 1$$

Moreover, we have

$$p_1 \times p_2(j_1 \times j_2, 0 | k_1 \times k_2) = p_1(j_1, 0 | k_1)p_2(j_2, 0 | k_2) = \delta_{j_1, k_1}\delta_{j_2, k_2} = \delta_{j_1 \times j_2, k_1 \times k_2}$$

(b) We show that the quantum system corresponding to the CSP of (a) is $L_2(K_1, \mu_1) \otimes L_2(K_2, \mu_2)$. Define the unitary operators on $L_2(K_1, \mu_1)$ and $L_2(K_2, \mu_2)$ given by

$$U_{rt}(j_r, k_r) = e^{i\theta_r(t, j_r, k_r)} p(j_r, t | k_r), \quad r = 1, 2$$

We then have

$$\begin{aligned} \text{tr}(U_{1t}^* \otimes U_{2t}^* P_{1j} \times P_{2j} U_{1t} \otimes U_{2t} P_{1k} \otimes P_{2k}) &= \text{tr}(U_{1t}^* P_{1j} U_{1t} P_{1k}) \text{tr}(U_{2t}^* P_{2j} U_{2t} P_{2k}) \\ &= p_1(j_1, t | k_1) p_2(j_2, t | k_2) = p_1 \times p_2(j_1 \times j_2 | k_1 \times k_2) \end{aligned}$$

Also,

$$\begin{aligned} &\text{tr}\left(U_{1t}^* \otimes U_{2t}^* P_{1j} \otimes P_{2j} U_{1t} \otimes U_{2t} \sum_{k_1, k_2=1}^{\infty} \mu_1 \times \mu_2(k_1 \times k_2) P_{1k_1} \otimes P_{2k_2}\right) \\ &= \text{tr}\left(U_{1t}^* P_{1j} U_{1t} \sum_{k_1=1}^{\infty} \mu_1(k_1) P_{1k_1}\right) \text{tr}\left(U_{2t}^* P_{2j} U_{2t} \sum_{k_2=1}^{\infty} \mu_2(k_2) P_{2k_2}\right) \\ &= p_1(j_1) p_2(j_2) = p_1 \times p_2(j_1 \times j_2) \quad \square \end{aligned}$$

Finally, let $L_2(K_1, \mu_1)$, $L_2(K_2, \mu_2)$ be two quantum systems and let $\rho_t^r = \sum_{k_r=1}^{\infty} \mu_r(k_r) U_{1t}^r P_{k_r}^r U_{1t}^{r*}$, $r = 1, 2$ be their induced quantum states. Then the induced quantum state for $L_2(K_1, \mu_1) \otimes L_2(K_2, \mu_2)$ is given by

$$\begin{aligned} \rho_t^1 \otimes \rho_t^2 &= \sum_{k_1=1}^{\infty} \mu_1(k_1) U_{1t}^1 P_{k_1}^1 U_{1t}^{1*} \otimes \sum_{k_2=1}^{\infty} \mu_2(k_2) U_{2t}^2 P_{k_2}^2 U_{2t}^{2*} \\ &= \sum_{k_1, k_2=1}^{\infty} \mu_1 \times \mu_2(k_1 \times k_2) (U_{1t}^1 \otimes U_{2t}^2) (P_{k_1}^1 \otimes P_{k_2}^2) (U_{1t}^1 \otimes U_{2t}^2)^* \end{aligned}$$

which is the induced state corresponding to the CSP $(K_1 \times K_2, p_1 \times p_2, \mu_1 \times \mu_2)$.

5 Classical Stochastic Processes

We have used CSP (K, p, μ) to derive a quantum system. The purpose of this exercise is that a CSP is defined in terms of real numbers and satisfies three simple axioms (1), (2). In contrast, quantum mechanics employs complex numbers and is based on non-intuitive mathematical concepts such as Hilbert spaces, self-adjoint and unitary operators traces and inner products. However, it should be pointed out that a CSP is not really a classical probabilistic concept. This is because a classical stochastic process need not satisfy the condition

$$\sum_{k=1}^{\infty} p(j, t | k) = 1 \quad (7)$$

To understand why, we define a *classical stochastic process* to be a 4-tuple (K, p, μ, f_t) , where, as before, $K = \{1, 2, \dots\}$ and μ is a nontrivial probability measure on K . The functions $f_t: K \rightarrow K$, $t \in \mathbb{R}$ are random variables and $f_t(k)$ is the system's configuration at time t when its configuration at initial time 0 is k . As before, $p: K \times \mathbb{R} \times K \rightarrow [0, 1]$ but we write $p(j, t \parallel k)$ to distinguish it from the CSP function $p(j, t \mid k)$. We define $p(j, t \parallel k)$ to be the classical conditional probability that the system is at configuration j at time t given it is at configuration k at initial time 0. This probability is given classically by

$$\begin{aligned} p(j, t \parallel k) &= \frac{\mu[\{r \in K: f_t(r) = j\} \cap \{r: f_0(r) = k\}]}{\mu[\{r: f_0(r) = k\}]} \\ &= \frac{\mu[f_t^{-1}(j) \cap f_0^{-1}(k)]}{\mu[f_0^{-1}(k)]} \\ &= \frac{\mu[f_t^{-1}(j) \cap \{k\}]}{\mu[\{k\}]} \end{aligned} \quad (8)$$

where the last equality comes from $f_0^{-1}(k) = k$. The properties of $p(j, t \parallel k)$ are given in the next theorem.

Theorem 3. The classical conditional probability satisfies the following:

- (a) $\sum_{j=1}^{\infty} p(j, t \parallel k) = 1$ for all $k \in K$, $t \in \mathbb{R}$,
- (b) $p(j, 0 \parallel k) = \delta_{jk}$ for all $j, k \in K$.
- (c) $\sum_{k=1}^{\infty} p(j, t \parallel k) = |f_t^{-1}(j)| = \text{cardinality of } f_t^{-1}(j)$ for all $j \in K$, $t \in \mathbb{R}$.

Proof. (a) Applying (8) we have for all $k \in K$, $t \in \mathbb{R}$

$$\begin{aligned} \sum_{j=1}^{\infty} p(j, t \parallel k) &= \frac{1}{\mu[\{k\}]} \sum_{j=1}^{\infty} \mu[f_t^{-1}(j) \cap \{k\}] \\ &= \frac{1}{\mu[\{k\}]} \mu[K \cap \{k\}] = 1 \end{aligned}$$

(b) By (8) we obtain for all $j, k \in \mathbb{R}$

$$p(j, 0 \parallel k) = \frac{\mu[f_0^{-1}(j) \cap \{k\}]}{\mu[\{k\}]} = \frac{\mu[\{j\} \cap \{k\}]}{\mu[\{k\}]} = \delta_{jk}$$

(c) By (8) we obtain for all $j \in K$, $t \in \mathbb{R}$

$$\begin{aligned} \sum_{k=1}^{\infty} p(j, t \parallel k) &= \sum_{k=1}^{\infty} \frac{\mu[f_t^{-1}(j) \cap \{k\}]}{\mu[\{k\}]} \\ &= \sum_{k \in f_t^{-1}(j)} \frac{\mu[\{k\}]}{\mu[\{k\}]} \\ &= \sum_{k \in f_t^{-1}(j)} 1 = |f_t^{-1}(j)| \quad \square \end{aligned}$$

We conclude that although p satisfies the usual conditions given by Theorem 3 (a), (b), it satisfies

$$\sum_{k=1}^{\infty} p(j, t \parallel k) = 1$$

if and only if $|f_t^{-1}(j)| = 1$ for all $j \in K$ which is equivalent to f_t being a bijection.

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