

Geometric Algebras and Fermion Quantum Field Theory

Stan Gudder

Department of Mathematics, University of Denver, Denver, Colorado, USA. E-mail: sgudder@du.edu

Editors: James F. Glazebrook & Danko D. Georgiev

Article history: Submitted on March 3, 2025; Accepted on July 20, 2025; Published on July 26, 2025.

Corresponding to a finite dimensional Hilbert space H with $\dim H = n$, we define a geometric algebra $\mathcal{G}(H)$ with $\dim [\mathcal{G}(H)] = 2^n$. The algebra $\mathcal{G}(H)$ is a Hilbert space that contains H as a subspace. We interpret the unit vectors of H as states of individual fermions of the same type and $\mathcal{G}(H)$ as a fermion quantum field whose unit vectors represent states of collections of interacting fermions. We discuss creation operators on $\mathcal{G}(H)$ and provide their matrix representations. Evolution operators provided by self-adjoint Hamiltonians on H and $\mathcal{G}(H)$ are considered. Boson-Fermion quantum fields are constructed. Extensions of operators from H to $\mathcal{G}(H)$ are studied. Finally, we present a generalization of our work to infinite dimensional separable Hilbert spaces.

Quanta 2025; 14: 48–65.

1 Basic Definitions and Preliminary Results

Unless stated otherwise, all vector spaces are complex and finite dimensional. Although the next three lemmas are known, we include their proofs for completeness.

Lemma 1. Let V be a vector space with basis f_1, f_2, \dots, f_n . For $a, b \in V$ with $a = \sum \alpha_i f_i$, $b = \sum \beta_i f_i$,

 This is an open access article distributed under the terms of the Creative Commons Attribution License CC-BY-3.0, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

$\alpha_i, \beta_i \in \mathbb{C}$, $i = 1, 2, \dots, n$, define $\langle a, b \rangle = \sum \bar{\alpha}_i \beta_i$. Then $(V, \langle \cdot, \cdot \rangle)$ is a complex inner product space.

Proof. If $\alpha \in \mathbb{C}$, then

$$\begin{aligned}\langle a, ab \rangle &= \sum \bar{\alpha}_i (\alpha b_i) = \alpha \sum \bar{\alpha}_i b_i = \alpha \langle a, b \rangle \\ \langle a, b \rangle &= \sum \bar{\alpha}_i \beta_i = \overline{\sum \alpha_i \bar{\beta}_i} = \overline{\langle b, a \rangle}\end{aligned}$$

If $c = \sum \gamma_i f_i$, then $a + b = \sum (\alpha_i + \beta_i) f_i$ and

$$\langle c, a + b \rangle = \sum \bar{\gamma}_i (\alpha_i + \beta_i) = \sum \bar{\gamma}_i \alpha_i + \sum \bar{\gamma}_i \beta_i = \langle c, a \rangle + \langle c, b \rangle$$

We also have

$$\langle a, a \rangle = \sum \bar{\alpha}_i \alpha_i = \sum |\alpha_i|^2 \geq 0$$

and $\langle a, a \rangle = 0$ if and only if $\alpha_i = 0$, $i = 1, 2, \dots, n$, which is equivalent to $a = 0$. \square

It follows that the vector space V of Lemma 1 is a Hilbert space with orthonormal basis f_1, f_2, \dots, f_n . We denote the set of linear operators on V by $\mathcal{L}(V)$. If $T \in \mathcal{L}(V)$ then $T f_j = \sum_k T_{kj} f_k$, $T_{kj} \in \mathbb{C}$ for all $k, j = 1, 2, \dots, n$. We say that the matrix $[T] = [T_{kj}]$ represents the operator T . Notice that

$$\langle f_k, T f_j \rangle = \left\langle f_k, \sum_i T_{ij} f_i \right\rangle = \sum_i T_{ij} \langle f_k, f_i \rangle = T_{kj}$$

so we can find T_{kj} explicitly.

Lemma 2. (i) If $[T_{kj}]$ represents T , then $\alpha [T_{kj}]$, $\alpha \in \mathbb{C}$, represents αT . (ii) If $[T_{kj}]$ represents T and $[S_{kj}]$ represents S , then $[T_{kj} + S_{kj}]$ represents $T + S$ and the usual matrix product $[T_{kj}] [S_{kj}]$ represents TS .

Proof. (i) This follows from

$$(\alpha T)f_j = \alpha Tf_j = \sum_k (\alpha T_{kj})f_k$$

for all $j = 1, 2, \dots, n$. (ii) Since

$$(T + S)f_j = Tf_j + Sf_j = \sum_k T_{kj}f_k + \sum_k S_{kj}f_k = \sum_k (T_{kj} + S_{kj})f_k$$

we have $[T_{kj} + S_{kj}]$ represents $T + S$. Since

$$(TS)f_j = T(Sf_j) = T\left(\sum_k S_{kj}f_k\right) = \sum_k S_{kj}Tf_k = \sum_k S_{kj}\left(\sum_i T_{ik}f_i\right) = \sum_{i,k} T_{ik}S_{kj}f_i = \sum_i ([T][S])_{ij}f_i$$

we have that $[T_{kj}][S_{kj}]$ represents TS . \square

If $T \in \mathcal{L}(V)$ we define the *adjoint* $T^* \in \mathcal{L}(V)$ by $\langle T^*a, b \rangle = \langle a, Tb \rangle$ for every $a, b \in V$.

Lemma 3. $S = T^*$ if and only if $\langle Sf_j, f_k \rangle = \langle f_j, Tf_k \rangle$ for all $j, k = 1, 2, \dots, n$.

Proof. If $S = T^*$, then clearly $\langle Sf_j, f_k \rangle = \langle f_j, Tf_k \rangle$ for all $j, k = 1, 2, \dots, n$. Conversely, suppose $\langle Sf_j, f_k \rangle = \langle f_j, Tf_k \rangle$ for all $j, k = 1, 2, \dots, n$. If $a = \sum \alpha_j f_j, b = \sum \beta_k f_k$, then

$$\begin{aligned} \langle Sa, b \rangle &= \left\langle S \sum \alpha_j f_j, \sum \beta_k f_k \right\rangle = \sum_{j,k} \bar{\alpha}_j \beta_k \langle Sf_j, f_k \rangle = \sum_{j,k} \bar{\alpha}_j \beta_k \langle f_j, Tf_k \rangle \\ &= \left\langle \sum \alpha_j f_j, T \sum \beta_k f_k \right\rangle = \langle a, Tb \rangle = \langle T^*a, b \rangle \end{aligned}$$

so $S = T^*$. \square

We say $T \in \mathcal{L}(V)$ is *self-adjoint* if $T = T^*$. It follows from Lemma 3 that T is self-adjoint if and only if $\langle Tf_j, f_k \rangle = \langle f_j, Tf_k \rangle$ for all $j, k = 1, 2, \dots, n$. We denote the set of self-adjoint operators on V by $\mathcal{L}_S(V)$. If $S, T \in \mathcal{L}_S(V)$, we write $S \leq T$ if $\langle a, Sa \rangle \leq \langle a, Ta \rangle$ for all $a \in V$ and call $T \in \mathcal{L}_S(V)$ *positive* if $T \geq 0$ where 0 is the zero operator. We call $T \in \mathcal{L}_S(V)$ an *effect* if $0 \leq T \leq I$ where I is the identity operator and denote the set of effects by $\mathcal{E}(V)$. An operator $T \in \mathcal{L}_S(V)$ is a *projection* if $T = T^2$. It is well-known that projections are effects and we call projections *sharp effects*. The *trace* of $T \in \mathcal{L}(V)$ is $\text{tr}(T) = \sum \langle f_j, Tf_j \rangle$. We call $\rho \in \mathcal{L}_S(V)$ a *state* if $\rho \geq 0$ and $\text{tr}(\rho) = 1$. The set of states is denoted by $\mathcal{S}(V)$. Finally, an operator $T \in \mathcal{L}(V)$ is *unitary* if $TT^* = I$ or equivalently $T^* = T^{-1}$.

We think of a Hilbert space as a mathematical structure that describes a quantum mechanical system [1–3]. In order to understand why this is so, we need to discuss states and effects on V . A state $\rho \in \mathcal{S}(V)$ corresponds to the initial condition of a quantum system. An effect $A \in \mathcal{E}(V)$ corresponds to a *yes–no*(true–false) measurement or experiment on the quantum system [3–5]. If A results in the outcome *yes* when it is measured, we say that A *occurs* and otherwise, it *does not occur*. It can be shown that $0 \leq \text{tr}(\rho A) \leq 1$ and we call $\text{tr}(\rho A)$ the *probability* that A occurs in the state ρ . An *observable* on V is a finite set of effects $A = \{A_x : x \in \Omega_A\}$ where $\sum_{x \in \Omega_A} A_x = I$ [4, 5]. We call Ω_A the *outcome set* of A and when A is measured and the resulting outcome x is observed, we say that the effect A_x *occurs*. If A is measured and the system is in state ρ , we call $P_\rho^A(x) = \text{tr}(\rho A_x)$ the *probability distribution* of A . Since

$$\sum_{x \in \Omega_A} P_\rho^A(x) = \sum_{x \in \Omega_A} \text{tr}(\rho A_x) = \text{tr}\left(\rho \sum_{x \in \Omega_A} A_x\right) = \text{tr}(\rho I) = \text{tr}(\rho) = 1$$

we see that P_ρ^A is indeed a probability measure. There is a close connection between observables and self-adjoint operators. If $A = \{A_x : x \in \Omega_A\}$ is an observable and $\{\lambda_x : x \in \Omega_A\} \subseteq \mathbb{R}$ then $B = \sum_{x \in \Omega_A} \lambda_x A_x$ is a self-adjoint operator. Conversely, if $B \in \mathcal{L}(V)$ then by the spectral theorem [4, 5], there exist a finite number of sharp effects A_i and real numbers $\lambda_i, i = 1, 2, \dots, m$ such that $\sum A_i = I$ and $B = \sum \lambda_i A_i$. Hence, $A = \{A_i : i = 1, 2, \dots, m\}$ is an observable. There is also a close connection between self-adjoint operators and the dynamics of a quantum system. This is

because $T \in \mathcal{L}(V)$ is unitary if and only if there exists an $A \in \mathcal{L}_S(V)$ such that $T = e^{iA}$ [4, 5]. If A corresponds to the Hamiltonian of a quantum system then the unitary group $U_t = e^{iAt}$, $t = [0, \infty)$, describes the dynamics of the system, where t is the time.

A state ρ is *pure* if it is a one-dimensional projection. In this case, there is a unit vector $\psi \in V$ such that $\rho(a) = \langle \psi, a\psi \rangle$ for every $a \in \mathcal{E}(V)$ and we write $\rho = \rho_\psi$. Since any state ρ is an affine combination of pure states ($\rho = \sum \lambda_i \rho_i$, $\lambda_i \geq 0$, $\sum \lambda_i = 1$, ρ_i , pure) we shall mainly consider only pure states.

2 Geometric Algebras and Fermion Quantum Fields

We now show that if H is a complex Hilbert space that describes an individual fermion, then the geometric algebra $\mathcal{G}(H)$ over H results in a fermion quantum field theory. Our definition of $\mathcal{G}(H)$ differs from the usual algebra in the sense that $\mathcal{G}(H)$ is complex while the usual algebra is real [6–14]. Let $\dim H = n$ and let e_1, e_2, \dots, e_n be an orthonormal basis for H . The *geometric algebra* $\mathcal{G}(H)$ over H is the complex homogeneous, associative, distribution algebra containing H that has the basis consisting of the elements $1 \in \mathbb{C}$

$$\begin{aligned} & \{e_i : i = 1, 2, \dots, n\}, \{e_i e_j : i, j = 1, 2, \dots, i < j\} \\ & \{e_i e_j e_k : i, j, k = 1, 2, \dots, n, i < j < k\} \\ & \quad \vdots \\ & \{\widehat{e_1} e_2 \cdots e_n, e_1 \widehat{e_2} e_3 \cdots e_n, \dots, e_1 e_2 \cdots e_{n-1} \widehat{e_n}\} \\ & e_1 e_2 \cdots e_n = \mathcal{I} \end{aligned}$$

where $e_1 e_2 \cdots \widehat{e_i} \cdots e_n$ means that e_i is not present. There is one additional axiom for $\mathcal{G}(H)$, namely, if $u = \sum_{j=1}^n c_j e_j \in H$, then $uu = \sum_{j=1}^n c_j^2 \in \mathbb{C}$.

If $u = \sum_{j=1}^n c_j e_j$, we define $\tilde{u} = \sum_{j=1}^n \bar{c}_j e_j$. It is easy to check that

$$(\alpha u + \beta v)^\sim = \bar{\alpha} \tilde{u} + \bar{\beta} \tilde{v}$$

for all $\alpha, \beta \in \mathbb{C}$. If $v = \sum d_j e_j$, we obtain

$$uv + vu = (u + v)(u + v) - uu - vv = \sum_{j=1}^n (c_j + d_j)^2 - \sum_{j=1}^n c_j^2 - \sum_{j=1}^n d_j^2 = 2 \sum_{j=1}^n c_j d_j = 2 \langle \tilde{u}, v \rangle$$

Hence, $\tilde{u} \perp v$ if and only if $uv = -vu$. It also follows that if $j \neq k$, then

$$\langle \tilde{e}_j, e_k \rangle = \langle e_j, e_k \rangle = 0$$

so $e_j e_k = -e_k e_j$. Notice that $uu = \langle \tilde{u}, u \rangle$ and if $u = e_1 + ie_2$ we have the unusual situation that $u \neq 0$ but $uu = 0$. Finally, we have that $uu = \sum_{j=1}^n c_j^2$ for all $u \in H$ if and only if $e_j e_j = 1$ and $e_j e_k = -e_k e_j$ for all $j \neq k$.

An element of the form $e_{i_1} e_{i_2} \cdots e_{i_r}$, $i_r \neq i_s$, is said to have *grade* j and grade $(1) = 0$. The set of linear combinations of grade j basis elements is a vector subspace of $\mathcal{G}(H)$ called the *grade j subspace* and is denoted $\mathcal{G}(H)_j$. By definition, 0 is considered to be every grade level because we want subspaces. Thus, $\mathcal{G}(H)_0 \approx \mathcal{G}(H)_n \approx \mathbb{C}$ and $\mathcal{G}(H)_1 = H$. We see that

$$\dim \mathcal{G}(H)_j = \binom{n}{j} = \frac{n!}{j!(n-j)!}$$

Hence, $\dim \mathcal{G}(H)_0 = \dim \mathcal{G}(H)_n = 1$ and by the binomial formula we have

$$\dim \mathcal{G}(H) = \sum_{j=0}^n \dim \mathcal{G}(H)_j = \sum_{j=0}^n \binom{n}{j} = (1+1)^n = 2^n$$

For $J_k = \{j_1, j_2, \dots, j_k\}$ with $j_1 < j_2 < \dots < j_k, j_i \in \{1, 2, \dots, n\}$ we define $e_0 = 1$

$$e_{J_k} = e_{j_1} e_{j_2} \cdots e_{j_k} \in \mathcal{G}(H)_k$$

and define $\mathcal{J} = \{0, J_k : k = 1, 2, \dots, n\}$. We make $\mathcal{G}(H)$ into a Hilbert space by declaring $\{e_J : J \in \mathcal{J}\}$ to be an orthonormal basis for $\mathcal{G}(H)$. This follows from the next corollary of Lemma 1.

Corollary 4. $(\mathcal{G}(H), \langle \cdot, \cdot \rangle)$ is a Hilbert space with orthonormal basis $\{e_J : J \in \mathcal{J}\}$ and inner product $\langle a, b \rangle = \sum_{J \in \mathcal{J}} \bar{\alpha}_J \beta_J$ where $a = \sum_{J \in \mathcal{J}} \alpha_J e_J, b = \sum_{J \in \mathcal{J}} \beta_J e_J$.

As before, we denote the set of linear operators on $\mathcal{G}(H)$ by $\mathcal{L}(\mathcal{G}(H))$ and the discussion of Section 1 on operators applies. In particular, if $T \in \mathcal{L}(\mathcal{G}(H))$, then $T e_J = \sum_K T_{KJ} e_K, T_{KJ} \in \mathbb{C}$ for all $K, J \in \mathcal{J}$ and the matrix $[T] = [T_{KJ}]$ represents T . Moreover, Lemmas 2 and 3 hold. Since $\mathcal{G}(H)$ is an algebra that is also a Hilbert space, we call $\mathcal{G}(H)$ a *Hilbert algebra*.

We think of $\mathcal{G}(H)$ as a quantum field theory describing a finite number of fermions of the same type. A basis multi-vector $v = e_{i_1} e_{i_2} \cdots e_{i_k}$ represents a state for k fermions of the same type (k electrons or k protons or k neutrons, ...). The actual state is ρ_v but we shall frequently abuse the notation and call any unit vector $a \in \mathcal{G}(H)$ a state when we really mean ρ_a . The Pauli exclusion principle postulates that two fermions of the same type cannot exist in the same state. This holds in the $\mathcal{G}(H)$ framework because if they are in the same state $e_i \in H$, then the resulting state for the pair would be $e_i e_i = 1$ which we call the *vacuum state*. In this sense, the two particles annihilate each other. It is interesting that three particles in the same state $e_i e_i e_i$ reduces to a single particle in the state e_i .

We call the grade 0 subspace $\mathcal{G}(H)_0 = \mathbb{C}$ the *vacuum subspace*, the grade 1 subspace $\mathcal{G}(H)_1 = H$ the *one-fermion subspace*, ..., the grade j subspace $\mathcal{G}(H)_j$ the *j-fermion subspace*. The reason for this is that $\mathcal{G}(H)_0$ corresponds to the states in which no fermion is present, ..., $\mathcal{G}(H)_j$ the states in which j fermions are present. In general, we call e_i a *one-fermion state*, ..., $e_{i_1} e_{i_2} \cdots e_{i_j}$ a *j-fermion state*. We also have anti-fermions (anti-electrons, anti-protons, ...). We call $\tilde{e}_i = e_1 \cdots \widehat{e}_i \cdots e_n$ an *anti-fermion state*,

$$(e_i e_j)^\sim = e_1 \cdots \widehat{e}_i \cdots \widehat{e}_j \cdots e_n$$

a 2-anti-fermion state, etc. Notice that $\tilde{1} = \mathcal{I}$ and we call $\mathcal{G}(H)_n \approx \mathbb{C}$ the *anti-vacuum subspace*. A fermion and its corresponding anti-fermion annihilate each other to form the anti-vacuum state \mathcal{I} .

If $a \in \mathcal{G}(H)_j, \|a\| = 1$, we call ρ_a a *j-fermion state* and otherwise ρ_a is a *combination fermion state*. In general, if $a \in \mathcal{G}(H)$ with $\|a\| = 1$ and $A \in \mathcal{E}(\mathcal{G}(H))$, the probability that A occurs in the state ρ_a becomes

$$\begin{aligned} P_{\rho_a}(A) &= \text{tr}(\rho_a A) = \sum_{i \in \mathcal{J}} \langle e_i, \rho_a(A e_i) \rangle = \sum_{i \in \mathcal{J}} \langle e_i, \langle a, A e_i \rangle a \rangle \\ &= \sum_{i \in \mathcal{J}} \langle a, A e_i \rangle \langle e_i, a \rangle = \sum_{i \in \mathcal{J}} \langle A a, e_i \rangle \langle e_i, a \rangle \\ &= \langle A a, a \rangle = \langle a, A a \rangle \end{aligned}$$

If $a = \sum_{j \in \mathcal{J}} \alpha_j e_j$ and $\alpha = (\alpha_j : j \in \mathcal{J})$ is the complex vector, we have

$$\begin{aligned} P_{\rho_a}(A) &= \left\langle \sum_{j \in \mathcal{J}} \alpha_j e_j, A \sum_{k \in \mathcal{J}} \alpha_k e_k \right\rangle = \sum_{j, k \in \mathcal{J}} \bar{\alpha}_j \alpha_k \langle e_j, A e_k \rangle \\ &= \langle \alpha, [A] \alpha \rangle \end{aligned}$$

3 Creation Operators

If $B \in \mathcal{G}(H)$ we define $\bar{B} \in \mathcal{L}(\mathcal{G}(H))$ by $\bar{B}a = Ba$. Notice that $\overline{(\alpha B)} = \alpha \bar{B}$, $\overline{(A + B)} = \bar{A} + \bar{B}$ and $\overline{(AB)} = \bar{A} \bar{B}$ for all $A, B \in \mathcal{G}(H)$. A particular example is the *creation operator* for a fermion in the state e_i given by

$$C_{e_i}(a) = \bar{e}_i(a) = e_i a$$

The following lemma will be useful.

Lemma 5. $e_1e_2 \cdots e_j e_1 e_2 \cdots e_j = 1$ if $j = 1, 4, 5, 8, 9, 12, 13, \dots$ and $e_1e_2 \cdots e_j e_1 e_2 \cdots e_j = -1$ if $j = 2, 3, 6, 7, 10, 11, 14, 15, \dots$

Proof. Clearly $e_1e_1 = 1$ and we have $e_1e_2e_1e_2 = -e_2e_1e_1e_2 = -e_2e_2 = -1$. Continuing, we obtain

$$e_1e_2e_3e_1e_2e_3 = e_2e_3e_2e_3 = -1$$

by the previous case. For $j = 6$ we have

$$e_1e_2e_3e_4e_5e_6e_1e_2e_3e_4e_5e_6 = -e_2e_3e_4e_5e_2e_3e_4e_5 = e_3e_4e_5e_3e_4e_5 = -1$$

by the previous case. For $j = 7$ we have

$$e_1e_2 \cdots e_7 e_1 e_2 \cdots e_7 = e_2 e_3 \cdots e_7 e_2 e_3 \cdots e_7 = -1$$

by the previous case. This pattern continues. For $j = 4$, we have

$$e_1e_2e_3e_4e_1e_2e_3e_4 = -e_2e_3e_4e_2e_3e_4 = 1$$

by the $j = 3$ case. For $j = 5$, we have

$$e_1e_2e_3e_4e_5e_1e_2e_3e_4e_5 = e_2e_3e_4e_5e_2e_3e_4e_5 = 1$$

by the previous case. Again the pattern continues. \square

Theorem 6. (i) The creation operator C_{e_i} is self-adjoint and unitary.

(ii) For $J = \{j_1, j_2, \dots, j_r\} \in \mathcal{J}$, the operator \bar{e}_J is unitary and it is self-adjoint if and only if $r \in \{1, 4, 5, 8, 9, 12, 13, \dots\}$,

Proof. (i) For $J, K \in \mathcal{J}$ we have $\langle e_K, C_{e_i}e_J \rangle = 0$ unless $e_K = \pm e_i e_J$ and if $e_K = \pm e_i e_J$, then $\langle e_K, e_i e_J \rangle = \pm 1$. Similarly $\langle C_{e_i}e_K, e_J \rangle = 0$ unless, $e_J = \pm e_i e_K$ and if $e_J = \pm e_i e_K$, then $\langle C_{e_i}e_K, e_J \rangle = \pm 1$. Also, $e_K = e_i e_J$ if and only if $e_J = e_i e_K$ and $e_K = -e_i e_J$ if and only if $e_J = -e_i e_K$. We conclude that

$$\langle e_K, C_{e_i}e_J \rangle = \langle C_{e_i}e_K, e_J \rangle$$

for every e_J, e_K so $C_{e_i} = C_{e_i}^*$ and C_{e_i} is self-adjoint. To show that C_{e_i} is unitary, we have

$$C_{e_i}C_{e_i}e_J = e_i e_i e_J = e_J$$

for every $J \in \mathcal{J}$. Hence, $C_{e_i}C_{e_i}^* = C_{e_i}C_{e_i} = I$ so C_{e_i} is unitary. (ii) The operator \bar{e}_J is unitary because $\bar{e}_J = C_{j_1}C_{j_2} \cdots C_{j_r}$ and the product of unitary operators is unitary. We have that \bar{e}_J is self-adjoint if and only if

$$C_{j_1}C_{j_2} \cdots C_{j_r} = (C_{j_1}C_{j_2} \cdots C_{j_r})^* = C_{j_r}^*C_{j_{r-1}}^* \cdots C_{j_1}^* = C_{j_r}C_{j_{r-1}} \cdots C_{j_1}$$

This equality holds if and only if

$$(C_{j_1}C_{j_2} \cdots C_{j_r})^2 = C_{j_1}C_{j_2} \cdots C_{j_r}C_{j_r}C_{j_{r-1}} \cdots C_{j_1} = 1$$

The result follows from Lemma 5. \square

Example 1. Letting $H = \mathbb{C}^2$, the algebra $\mathcal{G}(H)$ is 4-dimensional with basis

$$\begin{cases} 1 & \text{grade 0} \\ e_1 e_2 & \text{grade 1} \\ \mathcal{I} = e_1 e_2 & \text{grade 2} \end{cases}$$

The creation operators C_{e_1}, C_{e_2} are given by $C_{e_1}(1) = e_1, C_{e_1}(e_1) = 1, C_{e_1}(e_2) = \mathcal{I}, C_{e_1}(\mathcal{I}) = e_2$ and $C_{e_2}(1) = e_2, C_{e_2}(e_1) = -e_1e_2 = -\mathcal{I}, C_{e_2}(e_2) = 1, C_{e_2}(\mathcal{I}) = -e_1$. The corresponding matrices are

$$M[C_{e_1}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad M[C_{e_2}] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

It is clear that these matrices are unitary and self-adjoint. The operator $\bar{\mathcal{I}}$ is given by $\bar{\mathcal{I}}(1) = \mathcal{I}, \bar{\mathcal{I}}(e_1) = -e_2, \bar{\mathcal{I}}(e_2) = e_1, \bar{\mathcal{I}}(\mathcal{I}) = -1$. The corresponding matrix is

$$M[\bar{\mathcal{I}}] = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

We conclude that $\bar{\mathcal{I}}$ is unitary but not self-adjoint as shown in Theorem 6(ii). \square

Example 2. Letting $H = \mathbb{C}^3$, the algebra $\mathcal{G}(H)$ is 8-dimensional with basis

$$\begin{cases} 1 & \text{grade 0} \\ e_1 e_2 e_3 & \text{grade 1} \\ e_1 e_2 e_1 e_3 e_2 e_3 & \text{grade 2} \\ e_1 e_2 e_3 = \mathcal{I} & \text{grade 3} \end{cases}$$

The creation operator C_{e_1} is given by $C_{e_1}(1) = e_1, C_{e_1}(e_1) = 1, C_{e_1}(e_2) = e_1e_2, C_{e_1}(e_3) = e_1e_3, C_{e_1}(e_1e_2) = e_2, C_{e_1}(e_1e_3) = e_3, C_{e_1}(e_2e_3) = \mathcal{I}, C_{e_1}(\mathcal{I}) = e_2e_3$. The corresponding matrix is

$$M[C_{e_1}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

which is unitary and self-adjoint. Also, $M[C_{e_2}], M[C_{e_3}]$ are similar and are unitary, self-adjoint. The operator $\bar{e_1e_2}$ satisfies: $\bar{e_1e_2}(1) = e_1e_2, \bar{e_1e_2}(e_1) = -e_2, \bar{e_1e_2}(e_2) = e_1, \bar{e_1e_2}(e_3) = \mathcal{I}, \bar{e_1e_2}(e_1e_2) = -1, \bar{e_1e_2}(e_1e_3) = -e_2e_3, \bar{e_1e_2}(e_2e_3) = e_1e_3, \bar{e_1e_2}(\mathcal{I}) = -e_3$. The corresponding matrix is

$$M[\bar{e_1e_2}] = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We conclude that $\bar{e_1e_2}$ is unitary but not self-adjoint as shown in Theorem 6(ii). \square

We now consider the eigenvalues and eigenvectors of C_{e_i} .

Theorem 7. The eigenvalues of C_{e_i} are ± 1 . The normalized eigenvectors for 1 are $\frac{1}{\sqrt{2}}(e_j + e_i e_j)$ where $i \notin J$ and the normalized eigenvectors for -1 are $\frac{1}{\sqrt{2}}(e_j - e_i e_j)$ where $i \notin J$ and the normalized eigenvectors for -1 are $\frac{1}{\sqrt{2}}(e_j - e_i e_j)$ where $i \notin J$. There are 2^{n-1} normalized eigenvectors for eigenvalue 1 and 2^{n-1} normalized eigenvectors for eigenvalue -1.

Proof. Since C_{e_i} is self-adjoint and unitary, the eigenvalues of C_{e_i} are real and have absolute value 1. Hence, the eigenvalues λ satisfy $\lambda = \pm 1$. If $i \notin J$ we have

$$C_{e_i}(e_J + e_i e_J) = e_i e_J + e_i e_i e_J = e_i e_J + e_J$$

Hence, $\frac{1}{\sqrt{2}}(e_J + e_i e_J)$ is a normalized eigenvector for eigenvalue 1 for all J with $i \notin J$. Notice, there are 2^{n-1} such eigenvectors. If $i \notin J$ we have

$$C_{e_i}(e_J - e_i e_J) = e_i e_J - e_i e_i e_J = e_i e_J - e_J = -(e_J - e_i e_J)$$

Hence, $\frac{1}{\sqrt{2}}(e_J - e_i e_J)$ is a normalized eigenvector for eigenvalue -1 for all J with $i \notin J$. Again, there are 2^{n-1} such eigenvectors. Since $\dim[\mathcal{G}(H)] = 2^n$ we have found all the eigenvectors. of C_{e_i} \square

Notice that when $i \notin J$ we have

$$\langle e_J + e_i e_J, e_J - e_i e_J \rangle = \langle e_J, e_J \rangle - \langle e_J, e_i e_J \rangle + \langle e_i e_J, e_J \rangle - \langle e_i e_J, e_i e_J \rangle = 0$$

as it should be because eigenvectors for different eigenvalues are orthogonal.

Example 3. According to Theorem 7, if $H = \mathbb{C}^2$ the eigenvectors of C_{e_i} in $\mathcal{G}(H)$ are as follows. The $J \in \mathcal{J}$ for which $1 \notin J$ are $J = \{0\}$ and $J = \{2\}$. The resulting eigenvectors for eigenvalue 1 are

$$\frac{1}{\sqrt{2}}(1 + e_1 1) = \frac{1}{\sqrt{2}}(1 + e_1), \quad \frac{1}{\sqrt{2}}(e_2 + e_1 e_2)$$

and the eigenvectors for eigenvalue -1 are

$$\frac{1}{\sqrt{2}}(1 - e_1 1) = \frac{1}{\sqrt{2}}(1 - e_1), \quad \frac{1}{\sqrt{2}}(e_2 - e_1 e_2)$$

The corresponding matrix representations for these vectors are

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Applying $M[C_{e_1}]$ to these vector representations verify they are eigenvectors of C_{e_1} for eigenvalues ± 1 . We next consider C_{e_2} . The $J \in \mathcal{J}$ for which $2 \notin J$ are $J = \{0\}$ and $J = \{1\}$. The resulting eigenvectors for eigenvalue 1 are

$$\frac{1}{\sqrt{2}}(1 + e_2 1) = \frac{1}{\sqrt{2}}(1 + e_2), \quad \frac{1}{\sqrt{2}}(e_1 + e_2 e_1) = \frac{1}{\sqrt{2}}(e_1 - e_1 e_2) = \frac{1}{\sqrt{2}}(e_1 - \mathcal{I})$$

and the eigenvectors for eigenvalue -1 are

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Applying $M[C_{e_2}]$ to these vector representations verify they are eigenvectors of C_{e_2} for eigenvalues ± 1 . \square

Example 4. We now consider the matrix representations for the eigenvectors of C_{e_1} in $\mathcal{G}(H)$ where $H = \mathbb{C}^3$. The $J \in \mathcal{J}$ for which $1 \notin J$ are $J = \{0\}, \{2\}, \{3\}, \{2, 3\}$ the resulting eigenvectors for eigenvalue 1 are

$$\frac{1}{\sqrt{2}}(1 + e_1), \quad \frac{1}{\sqrt{2}}(e_2 + e_1 e_2), \quad \frac{1}{\sqrt{2}}(e_3 + e_1 e_3), \quad \frac{1}{\sqrt{2}}(e_2 e_3 + \mathcal{I})$$

and the eigenvectors for eigenvalue -1 are

$$\frac{1}{\sqrt{2}}(1 - e_1), \quad \frac{1}{\sqrt{2}}(e_2 - e_1 e_2), \quad \frac{1}{\sqrt{2}}(e_3 - e_1 e_3), \quad \frac{1}{\sqrt{2}}(e_2 e_3 - \mathcal{I})$$

The corresponding matrix representations for these vectors are

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

As in Example 3, these vectors form an orthonormal basis for $\mathcal{G}(H)$. Applying $M[C_{e_1}]$ to these vector representations verify they are eigenvectors of C_{e_1} for eigenvalues ± 1 . Similar results hold for C_{e_2} and C_{e_3} . \square

The *anti-commutant* of two operator S, T is

$$\{S, T\} = ST + TS$$

Theorem 8. (i) If $e_1 \neq e_2$, then $\{C_{e_1}, C_{e_2}\} = 0$. (ii) The eigenvectors $\frac{1}{\sqrt{2}}(e_J + e_i e_J)$, $\frac{1}{\sqrt{2}}(e_J - e_i e_J)$, $i \notin J$ form an orthonormal basis for $\mathcal{G}(H)$.

Proof. (i) This follows from

$$C_{e_1} C_{e_2} a = C_{e_1} e_2 a = e_1 e_2 a = -e_2 e_1 a = -C_{e_2} C_{e_1} a$$

for all $a \in \mathcal{G}(H)$. (ii) By Theorem 7, there are 2^n vectors of this form. Since eigenvectors corresponding to distinct eigenvalues of self-adjoint operators are orthogonal the first and second types are mutually orthogonal. Since $i \notin J_1, J_2$, if $J_1 \neq J_2$ then the two terms $e_{J_1}, e_i e_{J_1}$ are different than the two terms $e_{J_2}, e_i e_{J_2}$. Hence, vectors of the first type are orthogonal to other vectors of the first type and similarly for vectors of the second type. It follows that these vectors form an orthonormal basis for $\mathcal{G}(H)$. \square

We now consider the creation operator $C_{e_1} \in \mathcal{L}_S(\mathcal{G}(\mathbb{C}^2))$ in more detail. The operator C_{e_2} will be similar. Let ψ_{+1}, ψ_{+2} be the normalized eigenvectors corresponding to eigenvalue 1 and ψ_{-1}, ψ_{-2} be the normalized eigenvectors corresponding to eigenvalue -1. Let $P_{\psi_{+1}}$ be the projection onto ψ_{+1} . Then

$$P_{\psi_{+1}} 1 = \langle \psi_{+1}, 1 \rangle \psi_{+1} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

and similarly

$$P_{\psi_{+1}} e_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_{\psi_{+1}} e_2 = P_{\psi_{+1}} \mathcal{I} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We conclude that

$$P_{\psi_{+1}} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In a similar way we have

$$P_{\psi_{+2}} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The projection onto the eigenspace for eigenvalue 1 becomes

$$P_+ = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We consider P_+ to be the sharp effect that occurs when a fermion in the state e_1 is created.

Now let $P_{\psi_{-1}}$ be the projection onto ψ_{-1} . Then

$$P_{\psi_{-1}} 1 = \langle \psi_{-1}, 1 \rangle \psi_{-1} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

and similarly

$$P_{\psi_{-1}} e_1 = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_{\psi_{-1}} e_2 = P_{\psi_{-1}} \mathcal{I} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We conclude that

$$P_{\psi_{-1}} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In a similar way we have

$$P_{\psi_{-2}} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The projection onto the eigenspace for eigenvalue -1 becomes

$$P_- = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

We consider P_- to be the sharp effect that occurs when a fermion in the state e_1 is annihilated. As expected we have $P_+ + P_- = \mathcal{I}$. If the system is initially in the vacuum state 1 then the probability that a fermion in the state e_1 is created becomes

$$P^1(\text{create } \mathcal{I}_1) = \langle 1, P_+ 1 \rangle = \frac{1}{2} \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\rangle$$

$$= \frac{1}{2} \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\rangle = \frac{1}{2}$$

In a similar way we obtain

$$\begin{aligned} P^1(\text{annihilate } e_1) &= P^{e_1}(\text{create } e_1) = P^{e_1}(\text{annihilate } e_1) \\ &= P^{e_2}(\text{create } e_1) = P^{e_2}(\text{annihilate } e_1) = P^T(\text{create } e_1) \\ &= P^T(\text{annihilate } e_1) = 1/2 \end{aligned}$$

More generally, suppose the system is in the state

$$\psi = \frac{\alpha 1 + \beta e_1}{\sqrt{|\alpha|^2 + |\beta|^2}}$$

Then the probability that a fermion in the state e_1 is created becomes

$$\begin{aligned} P^\psi(\text{create } e_1) &= \langle \psi, P_+ \psi \rangle = \frac{1}{2(|\alpha|^2 + |\beta|^2)} \left\langle \begin{bmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{bmatrix} \right\rangle \\ &= \frac{1}{2(|\alpha|^2 + |\beta|^2)} \left\langle \begin{bmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha + \beta \\ \alpha + \beta \\ 0 \\ 0 \end{bmatrix} \right\rangle = \frac{|\alpha + \beta|^2}{2(|\alpha|^2 + |\beta|^2)} \end{aligned}$$

In particular, if $\alpha = \beta = 1$ we have $P^\psi(\text{create } e_1) = 1$

An operator $A \in \mathcal{L}_S(\mathcal{G}(\mathbb{C}^2))$ is an e_1 -observable if it has the form

$$A = \lambda_1 P_{\psi_{+1}} + \lambda_2 P_{\psi_{+2}} + \lambda_3 P_{\psi_{-1}} + \lambda_4 P_{\psi_{-2}}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$. In particular C_{e_1} is an e_1 -observable with $\lambda_1 = \lambda_2 = 1, \lambda_3 = \lambda_4 = -1$. An e_1 -observable A has the same eigenvectors as C_{e_1} with corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Its general form is

$$A = \frac{1}{2} \begin{bmatrix} \lambda_1 + \lambda_3 & \lambda_1 - \lambda_3 & 0 & 0 \\ \lambda_1 - \lambda_3 & \lambda_1 + \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_2 + \lambda_4 & \lambda_2 - \lambda_4 \\ 0 & 0 & \lambda_2 - \lambda_4 & \lambda_2 + \lambda_4 \end{bmatrix}$$

If A is measured, its possible outcomes are $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and when the system is in the state ρ , its probability distribution is

$$\begin{aligned} P_\rho^A(\lambda_1) &= \text{tr}(\rho P_{\psi_{+1}}), & P_\rho^A(\lambda_2) &= \text{tr}(\rho P_{\psi_{+2}}) \\ P_\rho^A(\lambda_3) &= \text{tr}(\rho P_{\psi_{-1}}), & P_\rho^A(\lambda_4) &= \text{tr}(\rho P_{\psi_{-2}}) \end{aligned}$$

We now consider the 8-dimensional Hilbert algebra $\mathcal{G}(\mathbb{C}^3)$. We will establish a pattern that the reader will see carries over to higher dimensions. As before, we consider the creation operator $C_{e_1} \in \mathcal{L}_S(\mathcal{G}(\mathbb{C}^3))$ and the operators C_{e_2}, C_{e_3} will be similar. Let $\psi_{+1}, \psi_{+2}, \psi_{+3}, \psi_{+4}$ be the normalized eigenvectors corresponding to eigenvalue 1 and $\psi_{-1}, \psi_{-2}, \psi_{-3}, \psi_{-4}$ be the normalized eigenvectors corresponding to eigenvalue -1. The corresponding projection operators become

$$P_{\psi_{+1}} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{\psi_{+2}} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{\psi_{+3}} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{\psi_{+4}} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

The projection onto the eigenspace for eigenvalue 1 becomes

$$P_+ = P_{\psi_{+1}} + P_{\psi_{+2}} + P_{\psi_{+3}} + P_{\psi_{+4}}$$

The -1 projection operators are

$$P_{\psi_{-1}} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{\psi_{-2}} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{\psi_{-3}} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{\psi_{-4}} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

The projection for eigenvalue -1 is $P_- = P_{\psi_{-1}} + P_{\psi_{-2}} + P_{\psi_{-3}} + P_{\psi_{-4}}$.

4 Boson-Fermion Quantum Fields

We now briefly consider boson and general boson-fermion quantum field theories. Let K be an m -dimensional Hilbert space. The corresponding *r-boson Hilbert* space is the Fock space [1, 2]

$$H = \mathbb{C} \oplus K \oplus K^2 \oplus \cdots \oplus K^r$$

where $K^i = K \otimes K \otimes \cdots \otimes K$ (i factors) and unit vectors in K^i represent states for i bosons. The *vacuum space* is \mathbb{C} and we see that we have states for 0 to r bosons. Letting $k = \dim H$ we have that

$$\begin{aligned} k &= 1 + m + m^2 + \cdots + m^r = 1 + m(1 + m + m^2 + \cdots + m^{r-1}) \\ &= 1 + m(k - m^r) = 1 + mk - m^{r+1} \end{aligned}$$

Hence, $(m - 1)k = m^{r+1} - 1$ so $k = \frac{m^{r+1} - 1}{m - 1}$. The simplest nontrivial case is $m = 2, r = 1, k = 3$. We thus have one boson with two possible basis states b_1, b_2 and as we shall see there are three fermions. The next simplest case is $m = 2, r = 2, k = 7$. In this case we have two bosons with basis states $b_1, b_2, b_1 \otimes b_1, b_1 \otimes b_2, b_2 \otimes b_1, b_2 \otimes b_2$ and we shall see there are 7 fermions.

Corresponding to H we have the *boson-fermion quantum field* $\mathcal{G}(H)$. This quantum field has r bosons and $k = \dim H = \frac{m^{r+1} - 1}{m - 1}$ fermions. As we have seen, $\mathcal{G}(H)$ is a Hilbert algebra with dimension 2^k . We illustrate this quantum field for the two simple cases mentioned above. In the case $m = 2, r = 1, k = 3$ we have one boson and three fermions. The bosons and fermions can interact and $\mathcal{G}(H)$ has $2^3 = 8$ basis elements. The Hilbert space $H = \mathbb{C} \oplus K$ has three basis elements v, b_1, b_2 where v is the boson vacuum state and b_1, b_2 are boson states. We write the basis states for $\mathcal{G}(H)$ as

$$\bar{1}, \bar{v}, \bar{b}_1, \bar{b}_2, \bar{v}\bar{b}_1, \bar{v}\bar{b}_2, \bar{b}_1\bar{b}_2, \bar{I}$$

We interpret $\bar{1}$ as the fermion vacuum state, \bar{v} is a fermion that has not interacted with a boson, \bar{b}_i is a fermion that has interacted with a boson in state $b_i, i = 1, 2$, $\bar{v}\bar{b}_i$ represents two fermions the first of which does not interact with a boson and the second interacts with a boson in state $b_i, i = 1, 2$, $\bar{b}_1\bar{b}_2$ represents two fermions where the first interacts with a boson in state b_1 and the second interacts with a boson in state b_2 . Finally $\bar{I} = \bar{v}\bar{b}_1\bar{b}_2$ is the anti-vacuum state.

Of course, the case $m = 2, r = 2, k = 7$ is much more complicated because we have two bosons and 7 fermions. In this case $\mathcal{G}(H)$ has $2^7 = 128$ basis elements. The basis states for H are $v_i, b_1, b_2, b_1 \otimes b_1, b_1 \otimes b_2, b_2 \otimes b_1, b_2 \otimes b_2$ and the basis states for $\mathcal{G}(H)$ are

$$\begin{aligned} &\bar{1}, \bar{v}, \bar{b}_1, \bar{b}_2, \bar{b}_1 \otimes \bar{b}_1, \bar{b}_2 \otimes \bar{b}_1, \bar{b}_2 \otimes \bar{b}_2 \\ &\bar{v}\bar{b}_1, \bar{v}\bar{b}_2, \bar{v}(\bar{b}_1 \otimes \bar{b}_1), \bar{v}(\bar{b}_1 \otimes \bar{b}_2), \bar{v}(\bar{b}_2 \otimes \bar{b}_1), \bar{v}(\bar{b}_2 \otimes \bar{b}_2) \\ &\bar{b}_1\bar{b}_2, \bar{b}_1(\bar{b}_1 \otimes \bar{b}_1), \bar{b}_1(\bar{b}_1 \otimes \bar{b}_2), \bar{b}_1(\bar{b}_2 \otimes \bar{b}_1), \bar{b}_1(\bar{b}_2 \otimes \bar{b}_2) \\ &\bar{b}_2(\bar{b}_1 \otimes \bar{b}_1), \bar{b}_2(\bar{b}_1 \otimes \bar{b}_2), \bar{b}_2(\bar{b}_2 \otimes \bar{b}_1), \bar{b}_2(\bar{b}_2 \otimes \bar{b}_2) \\ &\overline{(b_1 \otimes b_1)(b_1 \otimes b_2)}, \overline{(b_1 \otimes b_1)(b_2 \otimes b_1)}, \overline{(b_1 \otimes b_1)(b_2 \otimes b_2)} \\ &\overline{(b_1 \otimes b_2)(b_2 \otimes b_1)}, \overline{(b_1 \otimes b_2)(b_2 \otimes b_2)}, \overline{(b_2 \otimes b_1)(b_2 \otimes b_2)} \\ &\bar{v}\bar{b}_1\bar{b}_2, \bar{v}\bar{b}_1(\bar{b}_1 \otimes \bar{b}_1), \dots \\ &\vdots \\ &\bar{v}\bar{b}_1\bar{b}_2\overline{(b_1 \otimes b_1)(b_1 \otimes b_2)(b_2 \otimes b_1)}, \dots \bar{b}_1\bar{b}_2\overline{(b_1 \otimes b_1)(b_1 \otimes b_2)(b_2 \otimes b_1)(b_2 \otimes b_2)}\bar{I} \end{aligned}$$

In this case, we interpret $\overline{b_1 \otimes b_1}$ as a fermion that interacts with two bosons both of which in the state $b_1, \overline{b_1}(\overline{b_1 \otimes b_1})$ represents two fermions, the first of which interacts with a boson in state b_1 and the second interacts with two bosons in the state $b_1 \otimes b_1, \overline{b_1} \overline{b_1} \overline{b_2}$ represents three fermions, the first of which interacts with no boson, the second interacts with a boson in state b_1 and the third interacts with a boson in state b_2 . Higher order cases get exponentially larger. For example, the case $m = 3, r = 2, k = 13$ with two bosons and 13 fermions gives $\mathcal{G}(H)$ with $2^{13} = 8,192$ basis elements.

5 Evolution Operators

An operator U on $\mathcal{G}(H)$ is unitary if and only if there exists a self-adjoint operator A on $\mathcal{G}(H)$ such that $U = e^{i\pi A}$ where the constant π is for convenience and is not necessary [4, 5]. We define the *evolution operator* $U_t = e^{i\pi t A}$, where $t \in [0, \infty)$ represents the time and A is called a *Hamiltonian* for the system [1, 2]. If ϕ is a state on $\mathcal{G}(H)$, then $U_t(\phi)$ gives the evolution of ϕ relative to the Hamiltonian A . If A has spectral representation $A = \sum \lambda_j P_j$, $\lambda_j \in \mathbb{R}$, then

$$U_t = e^{i\pi t A} = \sum_j e^{i\pi t \lambda_j} P_j = \sum_j [\cos(\pi t \lambda_j) + i \sin(\pi t \lambda_j)] P_j$$

For example, in $\mathcal{G}(\mathbb{C}^2)$, C_{e_1} is self-adjoint and using C_{e_1} as the Hamiltonian we have

$$U_t = e^{i\pi t C_{e_1}} = \frac{e^{i\pi t}}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \frac{e^{-i\pi t}}{2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\pi t) & i \sin(\pi t) & 0 & 0 \\ i \sin(\pi t) & \cos(\pi t) & 0 & 0 \\ 0 & 0 & \cos(\pi t) & i \sin(\pi t) \\ 0 & 0 & i \sin(\pi t) & \cos(\pi t) \end{bmatrix}$$

In particular, the states $1, e_1, e_2, \mathcal{I}$ evolve according to

$$\begin{aligned} U_t(1) &= \cos(\pi t)1 + i \sin(\pi t)e_1 \\ U_t(e_1) &= i \sin(\pi t)1 + \cos(\pi t)e_1 \\ U_t(e_2) &= \cos(\pi t)e_2 + i \sin(\pi t)\mathcal{I} \\ U_t(\mathcal{I}) &= i \sin(\pi t)\mathcal{I}_2 + \cos(\pi t)\mathcal{I} \end{aligned}$$

Another way to view this is to use the fact that C_{e_1} is unitary so $C_{e_1} = e^{i\pi A}$ and apply the Hamiltonian $A = \frac{-i}{\pi} \ln C_{e_1}$. Since $\ln(-1) = i\pi$ we have

$$A = \frac{-i}{\pi} \ln C_{e_1} = \frac{-i}{\pi} [\ln(1)P_+ + \ln(-1)P_-] = -\left(\frac{i}{\pi}\right)i\pi P_- = P_-$$

Hence, letting $U_t = e^{i\pi t A}$ we obtain

$$U_t = e^{i\pi t P_-} = e^{i\pi t} P_- = \frac{1}{2} \begin{bmatrix} 1 + e^{i\pi t} & 1 - e^{i\pi t} & 0 & 0 \\ 1 - e^{i\pi t} & 1 + e^{i\pi t} & 0 & 0 \\ 0 & 0 & 1 + e^{i\pi t} & 1 - e^{i\pi t} \\ 0 & 0 & 1 - e^{i\pi t} & 1 + e^{i\pi t} \end{bmatrix}$$

In this case, the states $1, e_1, e_2, \mathcal{I}$ evolve according to

$$\begin{aligned} U_t(1) &= \frac{1}{2}(1 + e^{i\pi t})1 + \frac{1}{2}(1 - e^{i\pi t})e_1 \\ U_t(e_1) &= \frac{1}{2}(1 - e^{i\pi t})1 + \frac{1}{2}(1 + e^{i\pi t})e_1 \\ U_t(e_2) &= \frac{1}{2}(1 + e^{i\pi t})e_2 + \frac{1}{2}(1 - e^{i\pi t})\mathcal{I} \\ U_t(\mathcal{I}) &= \frac{1}{2}(1 - e^{i\pi t})e_2 + \frac{1}{2}(1 + e^{i\pi t})\mathcal{I} \end{aligned}$$

Let $A = \lambda_1 P_{\psi_{+1}} + \lambda_2 P_{\psi_{+2}} + \lambda_3 P_{\psi_{-1}} + \lambda_4 P_{\psi_{-2}}$, $\lambda_i \in \mathbb{R}$ be a C_{e_1} observable in $\mathcal{G}(\mathbb{C}^2)$. The corresponding evolution operator is

$$\begin{aligned} U_t &= e^{i\pi t A} = e^{i\pi t \lambda_1} P_{\psi_{+1}} + e^{i\pi t \lambda_2} P_{\psi_{+2}} + e^{i\pi t \lambda_3} P_{\psi_{-1}} + e^{i\pi t \lambda_4} P_{\psi_{-2}} \\ &= \frac{e^{i\pi t \lambda_1}}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{e^{i\pi t \lambda_2}}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} + \frac{e^{i\pi t \lambda_3}}{2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{e^{i\pi t \lambda_4}}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{i\pi t \lambda_1 + e^{i\pi t \lambda_3}} & e^{i\pi t \lambda_1} - e^{-i\pi t \lambda_3} & 0 & 0 \\ e^{i\pi t \lambda_1} - e^{i\pi t \lambda_3} & e^{i\pi t \lambda_1} + e^{i\pi t \lambda_3} & 0 & 0 \\ 0 & 0 & e^{i\pi t \lambda_2} + e^{i\pi t \lambda_4} & e^{i\pi t \lambda_2} - e^{i\pi t \lambda_4} \\ 0 & 0 & e^{i\pi t \lambda_2} - e^{i\pi t \lambda_4} & e^{i\pi t \lambda_2} + e^{i\pi t \lambda_4} \end{bmatrix} \end{aligned}$$

The evolution of the states $1, e_1, e_2, \mathcal{I}$ are given by

$$\begin{aligned} U_t(1) &= \frac{1}{2}(e^{i\pi t \lambda_1} + e^{i\pi t \lambda_3})1 + \frac{1}{2}(e^{i\pi t \lambda_1} - e^{-i\pi t \lambda_3})e_1 \\ U_t(e_1) &= \frac{1}{2}(e^{i\pi t \lambda_1} - e^{-i\pi t \lambda_3})1 + \frac{1}{2}(e^{i\pi t \lambda_1} + e^{i\pi t \lambda_3})e_1 \\ U_t(e_2) &= \frac{1}{2}(e^{i\pi t \lambda_2} + e^{i\pi t \lambda_4})e_2 + \frac{1}{2}(e^{i\pi t \lambda_2} - e^{i\pi t \lambda_4})\mathcal{I} \\ U_t(\mathcal{I}) &= \frac{1}{2}(e^{i\pi t \lambda_2} - e^{i\pi t \lambda_4})e_2 + \frac{1}{2}(e^{i\pi t \lambda_2} + e^{i\pi t \lambda_4})\mathcal{I} \end{aligned}$$

We next consider the operator $\bar{\mathcal{I}}$ on $\mathcal{G}(\mathbb{C}^2)$. We know that $\bar{\mathcal{I}}$ is unitary and since

$$\begin{aligned} \bar{\mathcal{I}}(e_1 + ie_2) &= i(e_1 + ie_2), \quad \mathcal{I}(e_1 - ie_2) = -i(e_1 - ie_2) \\ \mathcal{I}(1 + i\mathcal{I}) &= -i(1 + i\mathcal{I}), \quad \mathcal{I}(1 - i\mathcal{I}) = i(1 - i\mathcal{I}) \end{aligned}$$

the eigenvalue i has eigenvectors $\frac{1}{\sqrt{2}}(e_1 + ie_2)$ and $\frac{1}{\sqrt{2}}(1 - i\mathcal{I})$ while the eigenvalue $-i$ has eigenvectors $\frac{1}{\sqrt{2}}(e_1 - ie_2)$ and $\frac{1}{\sqrt{2}}(1 + i\mathcal{I})$. The projection onto the eigenspace for i is

$$P_{(i)} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & i & 1 & 0 \\ -i & 0 & 0 & 1 \end{bmatrix}$$

and the projection onto the eigenspace for $-i$ is

$$P_{(-i)} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix}$$

We now find the Hamiltonian A for the operator $\bar{\mathcal{I}}$. Since $\ln(i) = \frac{\pi}{2}i$ and $\ln(-i) = -\frac{\pi}{2}i$ and $\bar{\mathcal{I}} = e^{i\pi A}$ we conclude that

$$A = \frac{-i}{\pi} \ln(\bar{\mathcal{I}}) = \frac{-1}{\pi} [\ln(i)P_{(i)} + \ln(-i)P_{(-i)}] = \frac{1}{2}P_{(i)} - \frac{1}{2}P_{(-i)}$$

The dynamics for $\bar{\mathcal{I}}$ becomes

$$\begin{aligned} U_t &= e^{i\pi t A} = e^{i\frac{\pi}{2}t} P_{(i)} + e^{-i\frac{\pi}{2}t} P_{(-i)} = (\cos \frac{\pi}{2}t + i \sin \frac{\pi}{2}t)P_{(i)} + (\cos \frac{\pi}{2}t - i \sin \frac{\pi}{2}t)P_{(-i)} \\ &= (\cos \frac{\pi}{2}t)I + i \sin \frac{\pi}{2}t [P_{(i)} - P_{(-i)}] = (\cos \frac{\pi}{2}t)I + \sin \frac{\pi}{2}t \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \cos \frac{\pi}{2}t & 0 & 0 & -\sin \frac{\pi}{2}t \\ 0 & \cos \frac{\pi}{2}t & \sin \frac{\pi}{2}t & 0 \\ 0 & -\sin \frac{\pi}{2}t & \cos \frac{\pi}{2}t & 0 \\ \sin \frac{\pi}{2}t & 0 & 0 & \cos \frac{\pi}{2}t \end{bmatrix}$$

The evolution for the states $1, e_1, e_2, \mathcal{I}$ are given by

$$\begin{aligned} U_t(1) &= (\cos \frac{\pi}{2}t)1 + (\sin \frac{\pi}{2}t)\mathcal{I} \\ U_t(e_1) &= (\cos \frac{\pi}{2}t)1 - (\sin \frac{\pi}{2}t)e_2 \\ U_t(e_2) &= (\sin \frac{\pi}{2}t)e_1 + (\cos \frac{\pi}{2}t)e_2 \\ U_t(\mathcal{I}) &= -(\sin \frac{\pi}{2}t)e_1 + (\cos \frac{\pi}{2}t)\mathcal{I} \end{aligned}$$

We would like to point out the similarity between the operator $\overline{\mathcal{I}}$ on $\mathcal{G}(\mathbb{C}^2)$ and the operator $\overline{e_1 e_2}$ on $\mathcal{G}(\mathbb{C}^3)$. The eigenvalues of $\overline{e_1 e_2}$ are i and $-i$ and the eigenvectors for i are $\frac{1}{\sqrt{2}}(e_1 + ie_2)$, $\frac{1}{\sqrt{2}}(e_3 - i\mathcal{I})$, $\frac{1}{\sqrt{2}}(1 - ie_1 e_2)$, $\frac{1}{\sqrt{2}}e_3(e_1 + ie_2)$ and the eigenvectors for $-i$ are $\frac{1}{\sqrt{2}}(e_1 - ie_2)$, $\frac{1}{\sqrt{2}}(e_3 + i\mathcal{I})$, $\frac{1}{\sqrt{2}}(1 + ie_1 e_2)$, $\frac{1}{\sqrt{2}}e_3(e_1 - ie_2)$. The dynamics for $\overline{e_1 e_2}$ are simpler but more complicated than that of \mathcal{I} .

6 Extension Operators

We now discuss extensions of operators from H to $\mathcal{G}(H)$. Let $\dim H = n$ and $B \in \mathcal{L}(H)$. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_i \in \mathbb{C}$ we define

$$\begin{aligned} B^\alpha(1) &= \alpha_1 1 \\ B^\alpha(e_i) &= B(e_i), i = 1, 2, \dots, n \\ B^\alpha(e_1 e_j) &= \alpha_2 B(e_i) e_j + \alpha_2 e_i B(e_j) \text{ for } i, j = 1, 2, \dots, n \text{ with } i < j \\ B^\alpha(e_i e_j e_k) &= \alpha_3 [B(e_i) e_j e_k + e_i B(e_j) e_k + e_i e_j B(e_k)] \\ &\quad \text{for } i, j, k = 1, 2, \dots, n \text{ with } i < j < k \\ &\quad \vdots \\ B^\alpha(e_{i_1} e_{i_2} \cdots e_{i_k}) &= \alpha_k [B(e_{i_1}) e_{i_2} \cdots e_{i_k} + e_{i_1} B(e_{i_2}) \cdots e_{i_k} + \cdots + e_{i_1} e_{i_2} \cdots B(e_{i_k})] \\ &\quad \text{for } i_1, i_2, \dots, i_k = 1, 2, \dots, n \text{ with } i_1 < i_2 < \cdots < i_k \\ &\quad \vdots \\ B^\alpha(\mathcal{I}) &= \alpha_n [B(e_1) e_2 \cdots e_n + \cdots + e_1 e_2 \cdots B(e_n)] \end{aligned}$$

and extend B^α to $\mathcal{G}(H)$ by linearity. Then $B^\alpha \in \mathcal{L}(\mathcal{G}(H))$ and we call B^α the α -extension of B . For example, if $\alpha_i = 0$ for $i = 1, 2, \dots, n$, we call B^α the *trivial extension* of B and if $\alpha_i = \frac{1}{i}$ for $i = 1, 2, \dots, n$, we call B^α the *simple extension* of B . Notice that $(\beta B)^\alpha = \beta B^\alpha$ for any $\beta \in \mathbb{C}$ and $(A + B)^\alpha = A^\alpha + B^\alpha$ for any $A, B \in \mathcal{L}(H)$. However, $(AB)^\alpha \neq A^\alpha B^\alpha$ in general. For example, let $P \in \mathcal{L}(H)$ be the projection onto e_1 . Then $P^\alpha(e_1 e_2) = \alpha_2 e_1 e_2$ and $P^\alpha P^\alpha(e_1 e_2) = \alpha_2^2 e_1 e_2 \neq P^\alpha(e_1 e_2) = (PP)^\alpha(e_1 e_2)$. This also shows that if P is a projection, then P^α need not be a projection. Letting $I \in \mathcal{L}(H)$ be the identity operator, we have $I^\alpha(e_{i_1} e_{i_2} \cdots e_{i_j}) = j\alpha_j e_{i_1} e_{i_2} \cdots e_{i_j}$. Hence, $I^\alpha \in \mathcal{L}(\mathcal{G}(H))$ is the identity operator if and only if $\alpha_j = \frac{1}{j}$ which is equivalent to I^α being a simple extension of I .

Theorem 9. If $B \in \mathcal{L}_S(H)$ and $\alpha_i \in \mathbb{R}$, then $B^\alpha \in \mathcal{L}_S(\mathcal{G}(H))$.

Proof. Let $B(e_i) = \sum_{j=1}^n B_{ij} e_j$, $i = 1, 2, \dots, n$. Since B is self-adjoint, we have $B_{ij} = \overline{B}_{ji}$. Now $\langle e_{i_1} \cdots e_{i_k}, e_r e_{j_1} \cdots e_{j_s} \rangle \neq 0$ if and only if $e_{i_1} \cdots e_{i_k} = \pm e_r e_{j_1} \cdots e_{j_s}$ and in both of these cases we have

$$\langle e_{e_1} \cdots e_{i_k}, e_r e_{j_1} \cdots e_{j_s} \rangle = \langle e_r e_{i_1} \cdots e_{i_k}, e_{j_1} \cdots e_{j_s} \rangle \tag{1}$$

We then obtain by (1) that

$$\langle e_{i_1} e_{i_2} \cdots e_{i_r}, B^\alpha e_{j_1} e_{j_2} \cdots e_{j_s} \rangle = \alpha_s [\langle e_{i_1} e_{i_2} \cdots e_{i_k}, B(e_{j_1}) e_{j_2} \cdots e_{j_s} \rangle + \cdots + \langle e_{i_1} e_{i_2} \cdots e_{i_k}, e_{j_1} \cdots e_{j_{s-1}} B(e_{j_s}) \rangle]$$

$$\begin{aligned}
&= \alpha_s \left[\left\langle e_{i_1} e_{i_2} \cdots e_{i_k}, \sum_t B_{j_1 t}(e_t) e_{j_2} \cdots e_{j_s} \right\rangle + \cdots + \left\langle e_{i_1} e_{i_2} \cdots e_{i_k}, e_{j_1} \cdots e_{j_{s-1}} \sum_t B_{j_s t}(e_t) \right\rangle \right] \\
&= \alpha_s \left[\sum_t B_{j_1 t} \langle e_{i_1} e_{i_2} \cdots e_{i_k}, e_t e_{j_2} \cdots e_{j_s} \rangle + \cdots + \sum_t B_{j_s t} \langle e_{i_1} e_{i_2} \cdots e_{i_k}, e_{j_1} \cdots e_{j_{s-1}} e_t \rangle \right] \\
&= \alpha_s \left[\left\langle \sum_t B_{j_1 t}(e_t) e_{i_1} \cdots e_{i_k}, e_{j_2} \cdots e_{j_s} \right\rangle + \cdots + \left\langle \sum_t B_{j_s t}(e_t) e_{i_1} e_{i_2} \cdots e_{i_k}, e_{j_1} e_{j_2} \cdots e_{j_{s-1}} \right\rangle \right] \\
&= \langle B^\alpha e_{i_1} e_{i_2} \cdots e_{i_r}, e_{j_1} e_{j_2} \cdots e_{j_s} \rangle
\end{aligned}$$

It follows that B^α is self-adjoint. \square

Example 5. We now illustrate the proof of Theorem 9 with the example $H = \mathbb{C}^3$. Let $\alpha = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbb{R}^3$ and let $B \in \mathcal{L}_S(H)$ with $B(e_i) = \sum_j B_{ij} e_j$ so that $B_{ij} = \overline{B}_{ji}$. Unlike the proof of Theorem 9, we treat the various cases individually. Clearly, $\langle e_1, B^\alpha e_2 \rangle = \langle B^\alpha e_1, e_2 \rangle$, $\langle e_1, B^\alpha e_1 e_2 \rangle = \langle B^\alpha e_1, e_1 e_2 \rangle = 0$, $\langle e_1, B^\alpha \mathcal{I} \rangle = \langle B^\alpha e_1, \mathcal{I} \rangle = 0$, $\langle 1, B^\alpha e_1 \rangle = \langle B^\alpha 1, e_1 \rangle = 0$. We also have

$$\langle e_1 e_2, B^\alpha \mathcal{I} \rangle = \alpha_3 [\langle e_1 e_2, B e_1 e_2 e_3 \rangle + \langle e_1 e_2, e_1 B e_2 e_3 \rangle + \langle e_1 e_2, e_1 e_2 B e_3 \rangle] = 0 = \langle B^\alpha e_1 e_2, \mathcal{I} \rangle$$

Moreover,

$$\begin{aligned}
\langle e_1 e_2, B^\alpha e_1 e_2 \rangle &= \alpha_2 [\langle e_1 e_2, B e_1 e_2 \rangle + \langle e_1 e_2, e_1 B e_2 \rangle] \\
&= \alpha_2 [\langle e_1 e_2, B_{11} e_1 e_2 + B_{12} e_2 e_2 + B_{13} e_3 e_2 \rangle] + \alpha_2 [\langle e_1 e_2, e_1 B_{21} e_1 + e_1 B_{22} e_2 + e_1 B_{23} e_3 \rangle] \\
&= \alpha_2 (B_{11} + B_{22}) = \overline{\alpha_2} (\overline{B}_{11} + \overline{B}_{22}) = \overline{\langle e_1 e_2, B^\alpha e_1 e_2 \rangle} = \langle B^\alpha e_1 e_2, e_1 e_2 \rangle
\end{aligned}$$

and finally

$$\begin{aligned}
\langle e_1 e_2, B^\alpha e_1 e_3 \rangle &= \alpha_2 [\langle e_1 e_2, B e_1 e_3 \rangle + \langle e_1 e_2, e_1 B e_3 \rangle] \\
&= \alpha_2 [\langle e_1 e_2, (B_{11} e_1 + B_{12} e_2 + B_{13} e_3) e_3 \rangle + \langle e_1 e_2, e_1 (B_{31} e_1 + B_{32} e_2 + B_{33} e_3) \rangle] \\
&= \alpha_2 B_{32} = \alpha_2 \overline{B}_{23} \\
&= \alpha_2 [\langle (B_{11} e_1 + B_{12} e_2 + B_{13} e_3) e_2, e_1 e_3 \rangle + \langle e_1 (B_{21} e_1 + B_{22} e_2 + B_{23} e_3), e_1 e_3 \rangle] \\
&= \alpha_2 [\langle B e_1 e_2, e_1 e_3 \rangle + \langle e_1 B e_2, e_1 e_3 \rangle] = \langle B^\alpha e_1 e_2, e_1 e_3 \rangle
\end{aligned}$$

\square

A great simplification occurs if $A \in \mathcal{L}(H)$ is diagonal with respect to the basis e_1, e_2, \dots, e_n . In this case $A = \sum_{i=1}^n \lambda_i P_i$ where $\lambda_i \in \mathbb{R}$ and P_i is the projection onto e_i , $i = 1, 2, \dots, n$. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ we obtain $A^\alpha \in \mathcal{L}_S(\mathcal{G}(H))$ with $A^\alpha(1) = \alpha_1$, $A^\alpha(e_i) = A(e_i) = \lambda_i e_i$, $i = 1, 2, \dots, n$,

$$\begin{aligned}
A^\alpha(e_i e_j) &= \alpha_2 [A(e_i) e_j + e_i A(e_j)] = \alpha_2 (\lambda_i + \lambda_j) e_i e_j \\
A^\alpha(e_i e_j e_k) &= \alpha_3 [A(e_i) e_j e_k + e_i A(e_j) e_k + e_i e_j A(e_k)] \\
&= \alpha_3 (\lambda_i + \lambda_j + \lambda_k) e_i e_j e_k \\
&\vdots \\
A^\alpha(\mathcal{I}) &= \alpha_n (\lambda_1 + \lambda_2 + \cdots + \lambda_n) e_1 e_2 \cdots e_n
\end{aligned}$$

The eigenvalues of A^α are α_1, λ_i , $i = 1, 2, \dots, n$, $\alpha_2(\lambda_i + \lambda_j)$, $i, j = 1, 2, \dots, n$, $\alpha_3(\lambda_i + \lambda_j + \lambda_k)$, $i, j, k = 1, 2, \dots, n$, $\dots, \alpha_n(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$. The corresponding eigenvectors are the basis $1, e_i, e_i e_j, e_i e_j e_k, \dots, \mathcal{I}$. Considering A^α to be the Hamiltonian for the system, the corresponding dynamics is given by

$$\begin{aligned}
U_t^\alpha(1) &= e^{i\pi\alpha_1 t} 1 \\
U_t^\alpha(e_j) &= e^{i\pi\lambda_j t} e_j \\
U_t^\alpha(e_r e_j) &= e^{i\pi\alpha_2(\lambda_r + \lambda_j)t} e_r e_j \\
U_t^\alpha(e_r e_j e_k) &= e^{i\pi\alpha_3(\lambda_r + \lambda_j + \lambda_k)t} e_r e_j e_k, \dots \\
U_t^\alpha(\mathcal{I}) &= e^{i\pi\alpha_n(\lambda_1 + \lambda_2 + \cdots + \lambda_n)t} \mathcal{I}
\end{aligned}$$

We close by showing that this work extends to infinite dimensional separable Hilbert spaces.

Theorem 10. Let H be a separable infinite dimensional Hilbert space with orthonormal basis e_1, e_2, \dots . Then there exists a unique separable infinite dimensional Hilbert geometric algebra $\mathcal{G}(H)$ with the following properties: (i) $H \subseteq \mathcal{G}(H)$, (ii) If $u \in H$, then $\langle \tilde{u}, u \rangle = uu$. (iii) $\mathcal{G}(H)$ has the orthonormal basis given by

$$\begin{aligned}
& 1 \\
& \{e_i : i = 1, 2, \dots\} \\
& \{e_i e_j : i < j, i, j = 1, 2, \dots\} \\
& \{e_i e_j e_k : i < j < k, i, j, k = 1, 2, \dots\} \\
& \{e_i e_j e_k : i < j < k, i, j, k = 1, 2, \dots\} \\
& \vdots \\
& \{e_{i_1} e_{i_2} \cdots e_{i_n} : i_1 < i_2 \cdots < i_n, i_1, i_2, \dots, i_n = 1, 2, \dots\} \\
& \vdots
\end{aligned}$$

Proof. Let H_n be the n -dimensional subspace generated by e_1, e_2, \dots, e_n , $n = 1, 2, \dots$. Then the 2^n -dimensional Hilbert geometric algebra $\mathcal{G}(H_n)$ exists [14] and $\mathcal{G}(H_n) \subseteq \mathcal{G}(H_{n+1})$, $n = 1, 2, \dots$. Let $\mathcal{G}_0(H) = \bigcup_{n=1}^{\infty} \mathcal{G}(H_n)$. For $a, b \in \mathcal{G}_0(H)$, we have $a, b \in \mathcal{G}(H_n)$ for some n and we define $\langle a, b \rangle = \langle a, b \rangle_n$ in $\mathcal{G}(H_n)$ in which case $\langle a, b \rangle$ does not depend on n . It is clear that $\langle \cdot, \cdot \rangle$ is an inner product so $\mathcal{G}_0(H)$ is an inner product space with orthonormal basis given by the elements listed in (iii). If $\mathcal{G}(H)$ is the completion of $\mathcal{G}_0(H)$, then $\mathcal{G}(H)$ is the smallest Hilbert space containing $\mathcal{G}_0(H)$. It follows that the listed elements in (iii) form an orthonormal basis for $\mathcal{G}(H)$. A sequence $a_i \in \mathcal{G}(H)$ is *Cauchy* if for any $\epsilon > 0$ there exists an integer N_ϵ such that $i, j \geq N_\epsilon$ implies $\|a_i - a_j\| < \epsilon$. We then have that $a \in \mathcal{G}(H)$ if and only if there exists a Cauchy sequence $a_i \in \mathcal{G}_0(H)$ such that $\lim_{i \rightarrow \infty} \|a_i - a\| = 0$ so $\lim_{i \rightarrow \infty} a_i = a$. To verify (i), letting $a \in H$ we have $a = \sum_{i=1}^{\infty} c_i e_i$, $c_i \in \mathbb{C}$. Then we have $a = \lim a_n = \sum_{i=1}^n c_i e_i$ where $a_n \in H_n \subseteq \mathcal{G}_0(H)$. Hence, $a \in \mathcal{G}(H)$ so (i) holds. We now show that $\mathcal{G}(H)$ is a geometric algebra. if $a, b \in \mathcal{G}(H)$ then there exist $a_n, b_n \in \mathcal{G}_0(H)$ such that $\lim a_n = a$, $\lim b_n = b$ and we can assume that $a_n, b_n \in \mathcal{G}(H_n)$. Letting $c_n = a_n b_n$ we have that $c_n \in \mathcal{G}(H_n)$ and

$$\begin{aligned}
\|c_n - c_m\| &= \|a_n b_n - a_m b_m\| \leq \|a_n b_n - a_n b_m\| + \|a_n b_m - a_m b_m\| \\
&= \|a_n(b_n - b_m)\| + \|(a_n - a_m)b_m\|
\end{aligned}$$

We can consider $c \rightarrow a_n c$ as a linear operator on $\mathcal{G}(H_n)$. Since $\mathcal{G}(H_n)$ is finite dimensional, this operator is bounded with norm $\|a_n\|$. Since $\lim a_n = a$ there exists a $K \in \mathbb{R}^+$ such that $\|a_n\| \leq K$ for every n and similarly $\|b_m\| \leq M$ for every m , Hence,

$$\|c_n - c_m\| \leq K \|b_n - b_m\| + M \|a_n - a_m\|$$

Therefore, c_n is a Cauchy sequence and we define the product on $\mathcal{G}(H)$ by

$$a \cdot b = \lim c_n = \lim a_n b_n$$

It follows that if $a, b \in \mathcal{G}_0(H)$, then $a, b \in \mathcal{G}(H_n)$ for some n and $a \cdot b = ab$ so the product $a \cdot b$ extends that on $\mathcal{G}_0(H)$. To verify (ii), suppose $u \in H$. Then there exist $u_n \in H_n \subseteq \mathcal{G}(H_n) \subseteq \mathcal{G}_0(H)$ with $\lim u_n = u$. Then

$$u \cdot u = \lim u_n u_n = \lim \langle \tilde{u}_n, u_n \rangle = \langle \tilde{u}, u \rangle$$

so (ii) holds. To show that $\mathcal{G}(H)$ is a geometric algebra, it is clear that $a \cdot b$ is homogeneous. To show associativity, if $a, b, c \in \mathcal{G}(H)$, there exists $a_n, b_n, c_n \in \mathcal{G}_0(H)$ such that $\lim a_n = a$, $\lim b_n = b$ and $\lim c_n = c$. We then have

$$a \cdot (b \cdot c) = \lim a_n \cdot (\lim b_n \cdot \lim c_n) = \lim a_n \cdot [lim(b_n c_n)] = \lim a_n b_n c_n = \lim a_n b_n \cdot \lim c_n = a \cdot b \cdot c_n = (a \cdot b) \cdot c$$

To show distributivity, we have

$$a \cdot (b + c) = \lim a_n \cdot [lim(b_n + c_n)] = \lim a_n(b_n + c_n) = \lim a_n b_n + \lim a_n c_n = a \cdot b + a \cdot c$$

It follows that $\mathcal{G}(H)$ is a geometric algebra satisfying (i), (ii) and (iii). The uniqueness of $\mathcal{G}(H)$ is clear. Finally, assuming the axiom of choice, it follows that a countable union of countable sets is countable. We conclude that the orthonormal basis listed in (iii) is countable so $\mathcal{G}(H)$ is separable. \square

References

- [1] P. Busch, M. Grabowski, P. J. Lahti. Operational Quantum Physics. Vol. 31 of *Lecture Notes in Physics Monographs*. Springer, Berlin, 1997. doi: [10.1007/978-3-540-49239-9](https://doi.org/10.1007/978-3-540-49239-9).
- [2] P. Busch, P. J. Lahti, P. Mittelstaedt. The Quantum Theory of Measurement. Vol. 2 of *Lecture Notes in Physics Monographs*. Springer, Berlin, 1996. doi: [10.1007/978-3-540-37205-9](https://doi.org/10.1007/978-3-540-37205-9).
- [3] K. Kraus. States, Effects, and Operations: Fundamental Notions of Quantum Theory. Vol. 190 of *Lecture Notes in Physics*. Springer, Berlin, 1983. doi: [10.1007/3-540-12732-1](https://doi.org/10.1007/3-540-12732-1).
- [4] T. Heinosaari, M. Ziman. The Mathematical Language of Quantum Theory: From Uncertainty to Entanglement. Cambridge University Press, Cambridge, 2012. doi: [10.1017/cbo9781139031103](https://doi.org/10.1017/cbo9781139031103).
- [5] M. A. Nielsen, I. L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, Cambridge, 2010. doi: [10.1017/cbo9780511976667](https://doi.org/10.1017/cbo9780511976667).
- [6] J. W. Arthur. Understanding Geometric Algebra for Electromagnetic Theory. The IEEE Press Series on Electromagnetic Wave Theory. Wiley, Hoboken, New Jersey, 2011. doi: [10.1002/9781118078549](https://doi.org/10.1002/9781118078549).
- [7] C. Doran, A. Lasenby. Geometric Algebra for Physicists. Cambridge University Press, Cambridge, 2003. doi: [10.1017/cbo9780511807497](https://doi.org/10.1017/cbo9780511807497).
- [8] L. Dorst, D. Fontijne, S. Mann. Geometric Algebra for Computer Science. Morgan Kaufmann, San Francisco, 2009.
- [9] L. Dorst. The inner products of geometric algebra. in: L. Dorst, C. Doran, J. Lasenby (Eds.), Applications of Geometric Algebra in Computer Science and Engineering. Birkhäuser, Boston, 2002. pp. 35–46. doi: [10.1007/978-1-4612-0089-5_2](https://doi.org/10.1007/978-1-4612-0089-5_2).
- [10] F. J. Dyson. Missed opportunities. *Bulletin of the American Mathematical Society* 1972; **78**(5):635–652. doi: [10.1090/S0002-9904-1972-12971-9](https://doi.org/10.1090/S0002-9904-1972-12971-9).
- [11] D. Hestenes, G. Sobczyk. Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics. Vol. 5 of *Fundamental Theories of Physics*. D. Reidel, Dordrecht, 1984. doi: [10.1007/978-94-009-6292-7](https://doi.org/10.1007/978-94-009-6292-7).
- [12] D. Hestenes. Oersted Medal Lecture 2002: Reforming the mathematical language of physics. *American Journal of Physics* 2003; **71**(2):104–121. doi: [10.1119/1.1522700](https://doi.org/10.1119/1.1522700).
- [13] D. Hestenes. Space-Time Algebra. Birkhäuser, Cham, Switzerland, 2015. doi: [10.1007/978-3-319-18413-5](https://doi.org/10.1007/978-3-319-18413-5).
- [14] A. Macdonald. A survey of geometric algebra and geometric calculus. *Advances in Applied Clifford Algebras* 2017; **27**(1):853–891. doi: [10.1007/s00006-016-0665-y](https://doi.org/10.1007/s00006-016-0665-y).